Parabola

Reference: Advanced Level Pure Mathematics by S.L. Green (4th Edition) Chapter V p.61 - p.71

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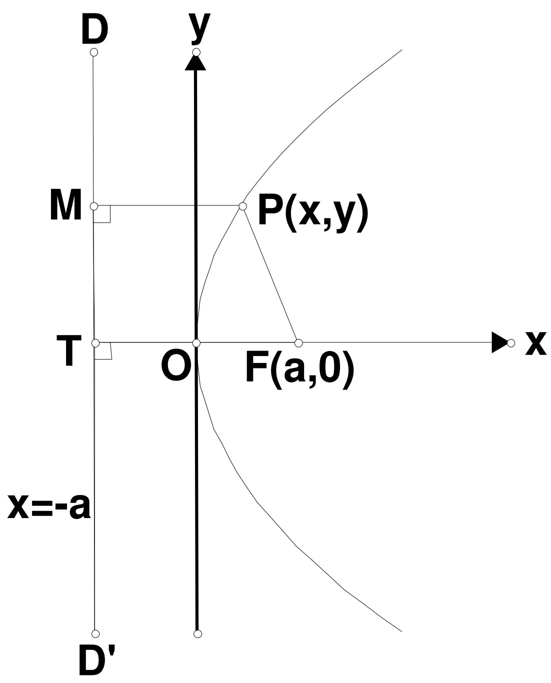
1. **Definition**: A **parabola** is defined as the locus of a point $P(x, y)$ moves in the x-y plane so that the distance from $P(x, y)$ to a fix point $F(a, 0)$ is equal to the distance from $P(x, y)$ to the straight line DD' ($x = -a$).
$F(a, 0)$ is called the **focus**, DD' ($x = -a$) is called the **directrix** of the curve.



2. **Equation**:
Let M be the foot of the perpendicular from P onto the directrix DD'. Then, by definition,
$PM = PF$

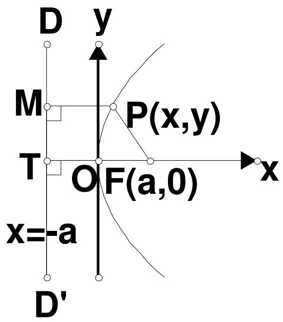
$$x + a = \sqrt{(x-a)^2 + y^2}$$

$$x^2 + 2ax + a^2 = x^2 - 2ax + a^2 + y^2$$

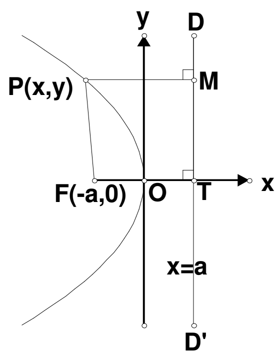
$$y^2 = 4ax$$

3. $O(0, 0)$ is called the **vertex** of the parabola. Obviously, $O(0, 0)$ lies on the parabola.
It is equal distance from $F(a, 0)$ and the directrix $x = -a$, i.e. $TO = OF$.
Replace y by $-y$ in $y^2 = 4ax$, there is no change. $\therefore$ The curve is **symmetrical about x-axis**, which is called the **axis** of the parabola.

4. **Variation**:
The constant "a" determines the direction of opening of the curve.
If $a > 0$, then it opens to the right. If $a < 0$, then it opens to the left.



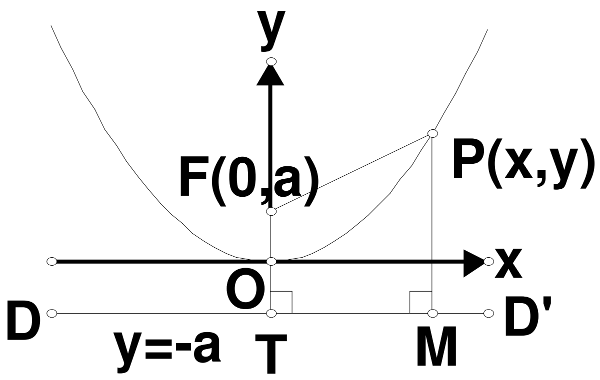
$a > 0$

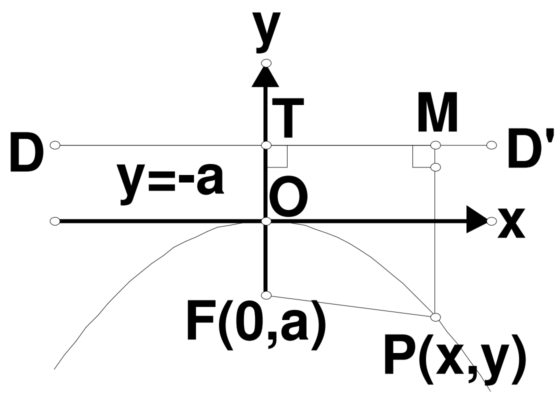


$a < 0$

Similarly $x^2 = 4ay$ describes a parabola which open upwards with symmetry axis $x = 0$, vertex = $O(0,0)$, focus $F = (0, a)$, directrix DD' is $y = -a$.

If $a > 0$, $x^2 = 4ay$ If $a < 0$, $x^2 = 4ay$





5. The **<u>latus rectum</u>** LL' is a line perpendicular to the axis of parabola drawn through the focus F

If $y^2 = 4ax$, then LL' is $x = a$

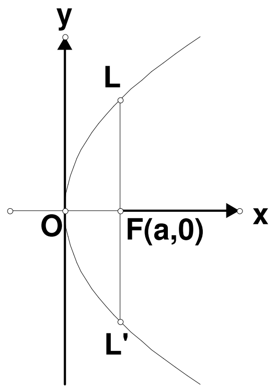
$$\begin{cases} y^2 = 4ax \\ x = a \end{cases}$$

$\Rightarrow y^2 = 4a^2$

$\Rightarrow y = 2a$ or $-2a$

$\therefore L = (a, 2a)$, $L' = (a, -2a)$

$LL' = \sqrt{(a-a)^2 + (2a+2a)^2} = |4a|$



6. If the vertex of the parabola is **translated** to a point $\boldsymbol{V(h, k)}$, then the equation of the parabola:

$(y - k)^2 = 4a(x - h)$

The new axis of the parabola is $y = k$

The new focus is $F(a + h, k)$

The new directrix is DD': $x = h - a$

Example Given $y = 4x^2 + 6x - 9$

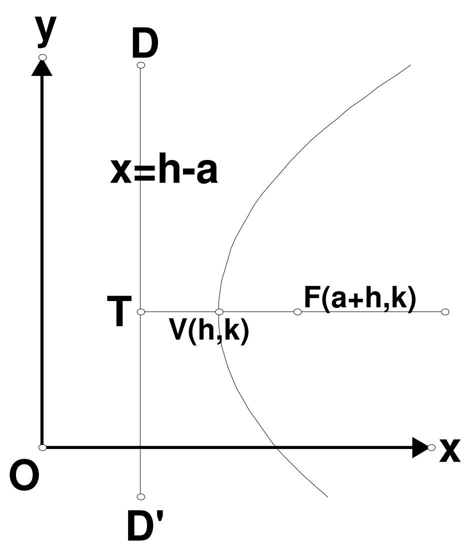
It can be transformed into the **<u>standard equation</u>** of a parabola by **<u>completing square</u>**:

$$\frac{y}{4} = x^2 + \frac{3}{2}x - \frac{9}{4}$$

$$\frac{y}{4} = x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - \frac{9}{4}$$

$$\frac{y}{4} = \left(x + \frac{3}{4}\right)^2 - \frac{45}{16}$$

$$4 \times \frac{1}{16}\left(y + \frac{45}{4}\right) = \left(x + \frac{3}{4}\right)^2$$



$a=\frac{1}{16}>0$, $V(h, k)=\left(-\frac{3}{4},-\frac{45}{4}\right)$

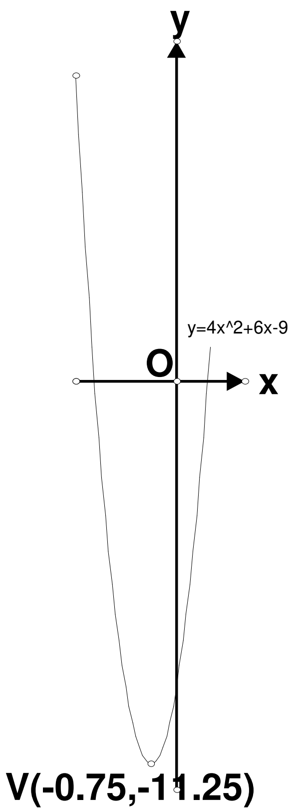
It is a parabola which **opens upwards**,

axis of parabola is $x+\frac{3}{4}=0$

The focus is at $\left(-\frac{3}{4},-\frac{45}{4}+\frac{1}{16}\right)=\left(-\frac{3}{4},-\frac{179}{16}\right)$

The directrix is at $y=-\frac{45}{4}-\frac{1}{16}=-\frac{181}{16}$

Latus Rectum LL' = $4\times\frac{1}{16}=\frac{1}{4}$



7. **Geometric Property**

Let P, Q be 2 points on the parabola $y^2 = 4ax$.

The chord PQ is produced to R on DD'.

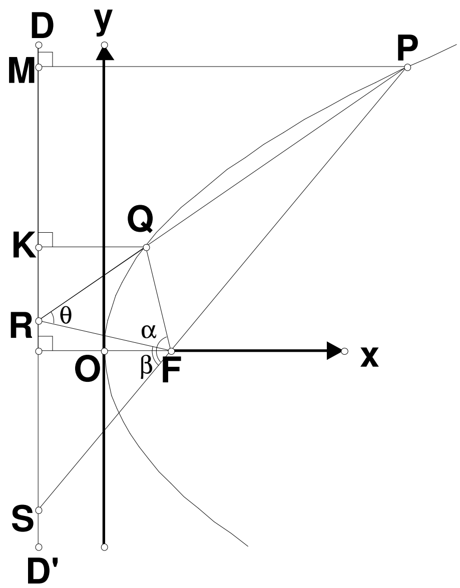
M is the foot of perpendicular drawn from P onto the line DD'.

K is the foot of perpendicular drawn from Q onto the line DD'.

Join RF, the line joining P, F is produced to meet DD' at S.

$PF = PM$ definition of parabola

$QF = QK$ definition of parabola



$\frac{PF}{QF}=\frac{PM}{QK}$ (1)

$\frac{PM}{QK}=\frac{PR}{QR}$ (2) $\because \Delta PMR \sim \Delta QKR$

Compare (1) and (2): $\frac{PF}{QF}=\frac{PR}{QR}$ (3)

Let $\angle QFR = \alpha$, $\angle RFS = \beta$, $\angle QRF = \theta$

By sine law, ΔFPR, $\frac{PF}{PR}=\frac{\sin\theta}{\sin\beta}$ ....(4)

ΔFQR, $\frac{QF}{QR}=\frac{\sin\theta}{\sin\alpha}$ ......(5)

By (4)÷(5) = (3) $\Rightarrow \alpha = \beta$

$\therefore$ ***RF*** is the **exterior angle bisector** of $\angle PFQ$.

8. <u>**Parametric Equation**</u>

$\begin{cases} x=at^2 \\ y=2at \end{cases}$ are the parametric equations of $y^2 = 4ax$, where t is the parameter.

9. <u>**Equation of chord**</u> (using parameters)

Let $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$ be 2 points on $y^2 = 4ax$.

Then the chord AB is: $\frac{y-2at_1}{x-at_1^2}=\frac{2at_2-2at_1}{at_2^2-at_1^2}$

After simplification, $x-\frac{1}{2}(t_1+t_2)y+at_1t_2=0$

10. **Tangent**

As $B(at_2^2, 2at_2)$ approaches $A(at_1^2, 2at_1)$, the chord AB becomes a tangent. Therefore the equation of tangent at $A(at_1^2, 2at_1)$ is $x - ty + at^2 = 0$

Let $P(x_0, y_0)$ be a point on the parabola $y^2 = 4ax$. Then $x_0 = at^2$, $y_0 = 2at$.

The equation of tangent is $x - ty + at^2 = 0$

$\Rightarrow x - \frac{y_0}{2a} y + x_0 = 0$

$2a(x + x_0) = y_0 y$

$y_0 y = 4a\left(\frac{x + x_0}{2}\right)$, this is the equation of tangent at (x_0, y_0)

It can be proved that the equation of tangent at (x_0, y_0) to $x^2 = 4ay$ is $x_0 x = 4a\left(\frac{y + y_0}{2}\right)$.

Or, in parametric form at $(2at, at^2)$: $y - tx + at^2 = 0$.

Suppose $y = Ax^2 + Bx + C$ is a parabola.

Let $P(x_0, y_0)$ be a point on the parabola.

The slope of tangent at $P(x_0, y_0)$ is given by $\frac{dy}{dx} = 2Ax_0 + B$

The equation of tangent is $\frac{y - y_0}{x - x_0} = 2Ax_0 + B$

$y - y_0 = (2Ax_0 + B)x - (2Ax_0 + B)x_0$

$y + y_0 = 2y_0 + (2Ax_0 + B)x - (2Ax_0 + B)x_0$

$\frac{1}{2}(y + y_0) = Ax_0^2 + Bx_0 + C + (Ax_0 + \frac{B}{2})x - (Ax_0 + \frac{B}{2})x_0$

$\frac{1}{2}(y + y_0) = (Ax_0 + \frac{B}{2})x + \frac{B}{2}x_0 + C$

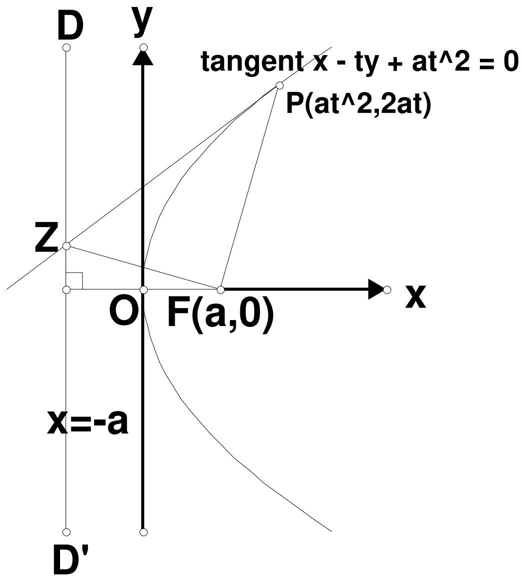
$\frac{1}{2}(y + y_0) = Ax_0 x + \frac{B}{2}(x + x_0) + C$

The rule is: change $y \to \frac{1}{2}(y + y_0)$, $x^2 \to x_0 x$, $x \to \frac{1}{2}(x + x_0)$ from $y = Ax^2 + Bx + C$.

11. **The part of tangent and the directrix subtends a right angle at the focus.**

Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$.

The tangent at P cuts the directrix DD' at Z, then $\angle PFZ = 90°$



Proof: To find Z: $\begin{cases} x - ty + at^2 = 0 \\ x + a = 0 \end{cases}$.

$-a - ty + at^2 = 0$

$ty = a(t^2 - 1)$

$y = \frac{a(t^2 - 1)}{t}$

$\therefore Z(-a, \frac{a(t^2 - 1)}{t})$

$m_{ZF} \times m_{PF} = \frac{a(t^2 - 1)}{t(-a - a)} \times \frac{2at}{at^2 - a}$

$= -\frac{a(t^2 - 1)}{2at} \times \frac{2at}{a(t^2 - 1)} = -1$

$\therefore ZF \perp PF$

12. If PT is a tangent at P, M is the foot of perpendicular from P on DD', then **PT bisect $\angle PMF$**.

In the figure, we want to prove that $\alpha = \beta$.

PT: $x - ty + at^2 = 0$

As PM is parallel to x-axis,

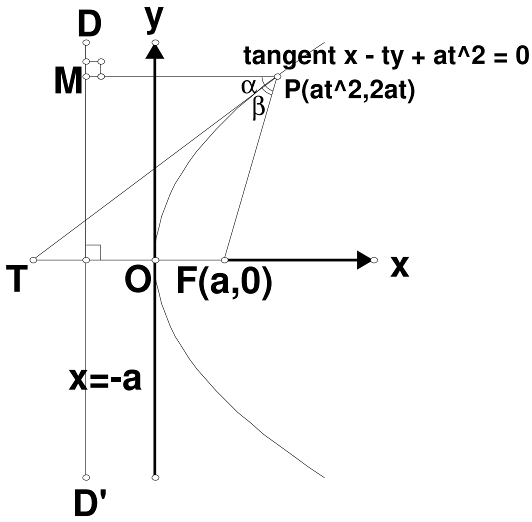
$\tan\alpha$ = slope of $PT = \frac{1}{t}$.

$$m_{PF} = \frac{2at}{at^2 - a} = \frac{2t}{t^2 - 1}$$

$$\tan\beta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{2t}{t^2-1} - \frac{1}{t}}{1 + \frac{2t}{t^2-1}\times\frac{1}{t}}$$

$$= \frac{2t^2 - t^2 + 1}{t^3 - t + 2t} = \frac{t^2+1}{t(t^2+1)} = \frac{1}{t}$$

$\therefore \tan\alpha = \tan\beta \Rightarrow \alpha = \beta$



13. **If the tangent at P cuts the x-axis at T, then $FP = TF$.**

PT: $x - ty + at^2 = 0$

T is given by letting $y = 0$

$x = -at^2$

$\therefore T(-at^2, 0)$

$TF = |a + at^2| = |a|(1 + t^2)$

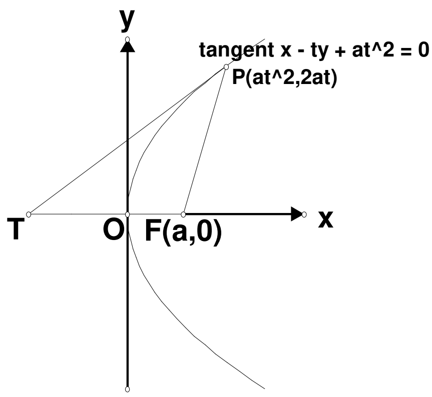
$$PF = \sqrt{(at^2 - a)^2 + (2at)^2}$$

$$= \sqrt{a^2(t^4 - 2t^2 + 1 + 4t^2)}$$

$$= \sqrt{a^2(t^4 + 2t^2 + 1)}$$

$= |a|(1 + t^2)$

$\therefore TF = PF$



14. **If the tangent at P cuts the y-axis at S, then $\angle PSF = 90°$.**

PS: $x - ty + at^2 = 0$

S is given by letting $x = 0$

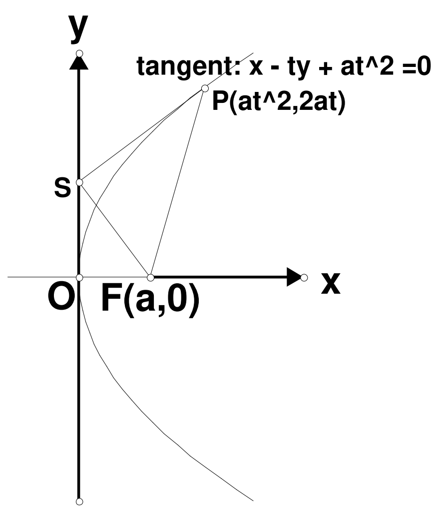
$-ty + at^2 = 0$

Assume $t \neq 0$, $y = at \Rightarrow S(0, at)$

$$m_{PS} \times m_{SF} = \frac{2at - at}{at^2} \times \frac{at}{-a}$$

$$= \frac{at}{at^2} \times \frac{t}{-1} = -1$$

$\therefore PS \perp SF$



Exercise 1 In the above section, prove that $SF^2 = |a| \times PF$

15. **Condition for tangency**. Let $\ell x + my + n = 0$ be a tangent.

Then $\ell x + my + n = 0$ is proportional to $x - ty + at^2 = 0$

$$\frac{\ell}{1} = \frac{m}{-t} = \frac{n}{at^2}$$

$$\Rightarrow t = -\frac{m}{\ell} = -\frac{n}{am}$$

$$\Rightarrow am^2 = n\ell$$

Let $y = mx + c$ be a tangent with a given slope m.

Then $mx - y + c = 0$ is proportional to $x - ty + at^2 = 0$

$$\frac{m}{1} = \frac{-1}{-t} = \frac{c}{at^2}$$

$$t = \frac{1}{m} = \frac{c}{a}$$

$$\Rightarrow c = \frac{a}{m}$$

$\therefore$ Given a slope m, the equation of tangent is: $y = mx + \frac{a}{m}$.

Example 1 Find the equation and the point of contact of the tangent to the parabola $y^2 = 8x$ which is parallel to $y = -3x$.

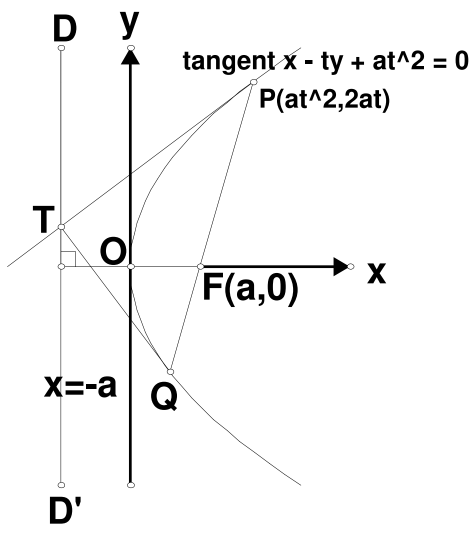
From above, $a = 2$, $m = -3$; the equation of tangent is $y = -3x - \frac{2}{3} \Rightarrow x + \frac{1}{3}y + \frac{2}{9} = 0$.

Compare it with $x - ty + at^2 = 0$, $\Rightarrow t = -\frac{1}{3}$.

$\therefore$The point of contact is $(at^2, 2at) = (\frac{2}{9}, -\frac{4}{3})$.

16. The line joining $P(at^2, 2at)$ and the focus F is produced to meet $y^2 = 4ax$ again at Q. Show that the tangent at P is perpendicular to the tangent at Q and they meet at the directrix.

Suppose the tangents at P and Q meet at T.



PF is given by $\frac{y}{x-a} = \frac{2at}{at^2 - a}$

$\frac{y}{x-a} = \frac{2t}{t^2-1} \Rightarrow (t^2 - 1)y = 2tx - 2at$

To find Q: let $Q = (at_1^2, 2at_1)$

$(t^2 - 1)(2at_1) = 2t(at_1^2) - 2at$

$(t^2 - 1)t_1 = tt_1^2 - t$

$t^2t_1 - t_1 - tt_1^2 + t = 0$

$tt_1(t - t_1) + (t - t_1) = 0$

$(t - t_1)(tt_1 + 1) = 0$

$\because P \neq Q$, $t \neq t_1$; $\therefore tt_1 + 1 = 0$

$t_1 = -\frac{1}{t}$

PT: $x - ty + at^2 = 0$

QT: $x - t_1y + at_1^2 = 0 \Rightarrow x + \frac{1}{t}y + \frac{a}{t^2} = 0$

$$\begin{cases} x - ty + at^2 = 0 \cdots\cdots(1) \\ t^2x + ty + a = 0 \cdots\cdots(2) \end{cases}$$

(1) + (2) $(1 + t^2)x + a(1 + t^2) = 0$

$(1 + t^2)(x + a) = 0$

$\because 1 + t^2 \neq 0 \therefore x + a = 0$, which is the directrix.

$\therefore PT$ and QT meet at the directrix.

$m_{PT} \times m_{QT} = \frac{1}{t} \times (-t)$

$= -1$

$\therefore PT \perp QT$

17. Given the mid-point $R(h, k)$, to find the equation of chord.

Suppose the chord through R cuts the parabola at $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$.

As R is the mid point of PQ,

$k = \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)$

$\Rightarrow t_1 + t_2 = \frac{k}{a}$

$m_{PQ} = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}$

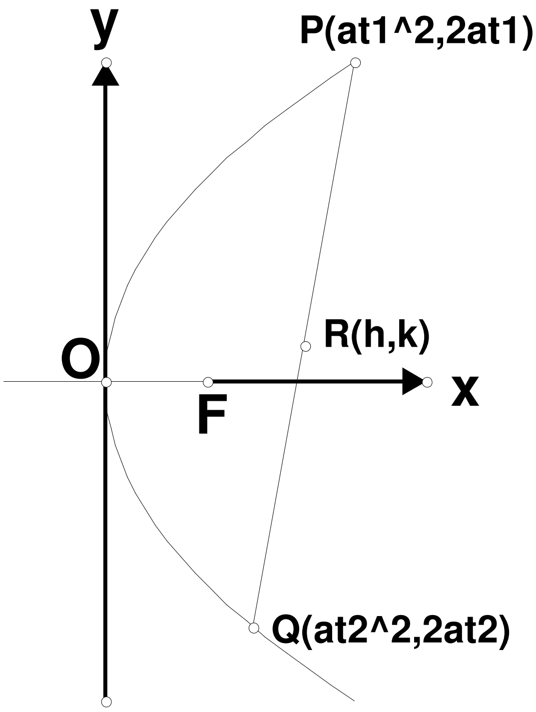
$= \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_1 + t_2}$

$\therefore m_{PQ} = \frac{2a}{k}$

$\therefore PQR$ is $y - k = \frac{2a}{k}(x - h)$

$ky - k^2 = 2ax - 2ah$

$2ax - ky + k^2 - 2ah = 0$, this is the equation of chord, given the mid-point $R(h, k)$



Example 2 Find the locus of the mid-points of chords which have slope = tan θ

Let $R(h, k)$ be the mid-point of the chord.

Then from the above result, $2ax - ky + k^2 - 2ah = 0$

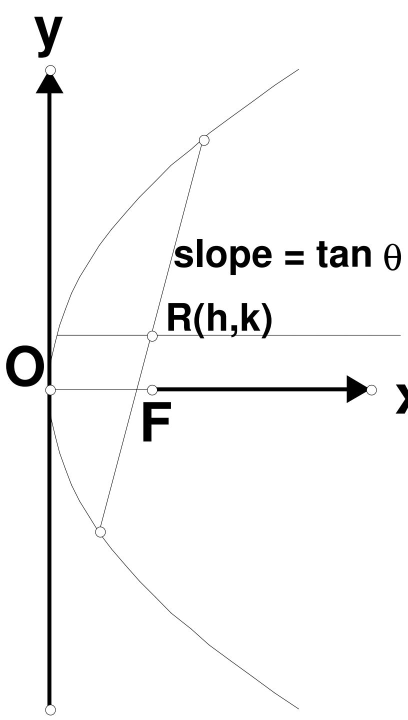
$\tan\theta = \frac{2a}{k}$

$\therefore k = 2a \cot\theta$

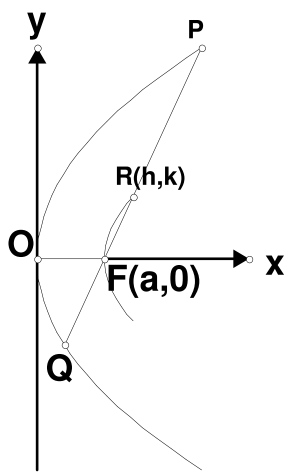
Change $(h, k) \rightarrow (x, y)$

The locus is $y = 2a \cot\theta$

It is parallel to x-axis, which is called the **<u>diameter</u>**

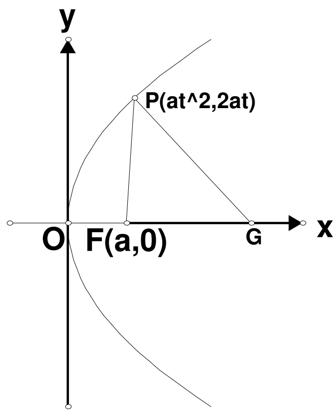


Example 3 Find the locus of the mid-points of a variable focal chord.
Let $R(h, k)$ be the mid-point of a focal chord.
Then from the above result, $2ax - ky + k^2 - 2ah = 0$
It passes through $F(a, 0)$
$\Rightarrow 2a^2 - 2ah + k^2 = 0$
Change $(h, k) \to (x, y)$
The locus is: $2a^2 - 2ax + y^2 = 0$
$y^2 = 2a(x - a)$ which is also a parabola.

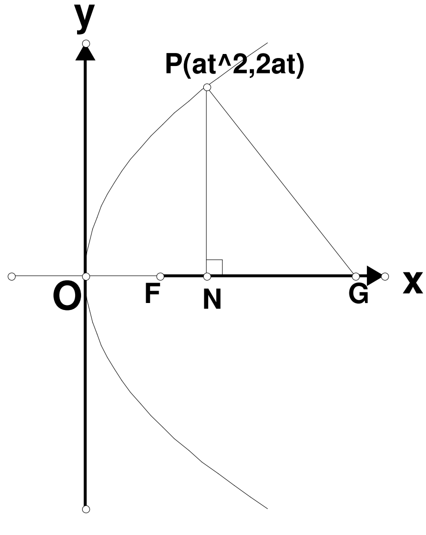


18. **Normal** at $P(at^2, 2at)$. It is a line which is perpendicular to the tangent at the point of contact.
The equation of tangent is $x - ty + at^2 = 0$
slope $= \frac{1}{t}$
$\therefore$ slope of the normal $= -t$
$\Rightarrow$ equation of the normal is $\frac{y - 2at}{x - at^2} = -t$
$\Rightarrow tx + y - 2at - at^3 = 0$

19. If the normal at $P(at^2, 2at)$ cuts the x-axis at G, then $PF = FG$
To find G, let $y = 0$
$tx - 2at - at^3 = 0$
$t \neq 0$, $x = 2a + at^2 \Rightarrow G(2a + at^2, 0)$
$PF = \sqrt{(at^2 - a)^2 + (2at)^2}$
$= |a|(t^2 + 1)$
$FG = |2a + at^2 - a| = |a|\ (t^2 + 1)$
$\therefore PF = FG$



20. If N is the foot of perpendicular from P onto x-axis, then $NG = 2|a|$
coordinates of $N = (at^2, 0)$, $G = (2a + at^2, 0)$
$NG = |2a + at^2 - at^2|$
$= 2|a|$



Example 4 Find the locus of the mid-point of a normal chord to $y^2 = 4ax$

Let the chord be $x - \frac{1}{2}(t_1 + t_2)y + at_1t_2 = 0$

It is a normal at $A(at_1^2, 2at_1)$: $t_1x + y - 2at_1 - at_1^3 = 0$

The two equations are proportional: $\frac{1}{t_1} = \frac{t_1 + t_2}{-2} = \frac{at_1t_2}{-a(2t_1 + t_1^3)}$

$\Rightarrow\ t_2 = -t_1 - \frac{2}{t_1}$ ....(1)

Let $M(x_0, y_0)$ be the mid-point of chord AB.

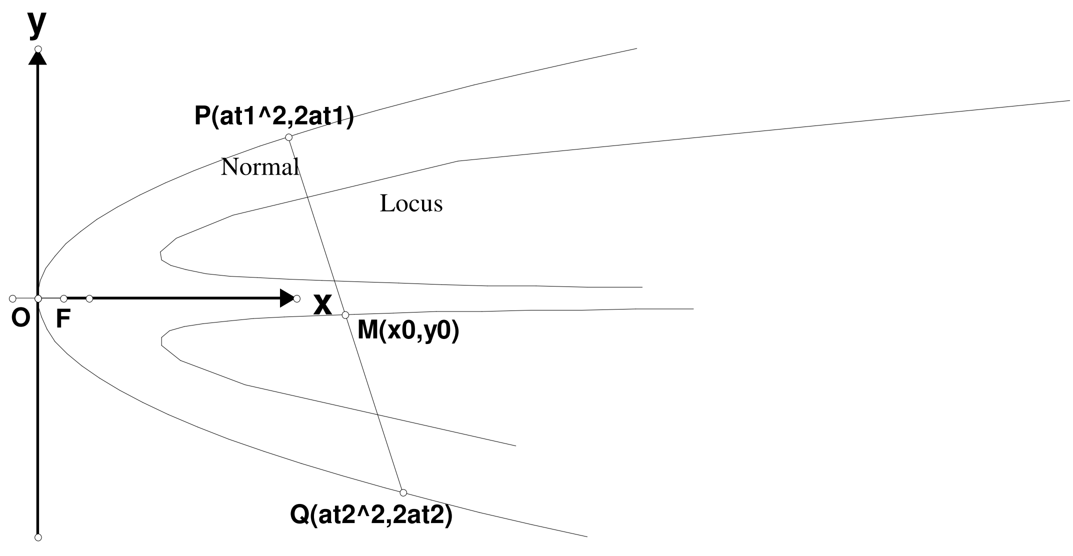
$x_0 = \frac{a(t_1^2 + t_2^2)}{2}$, $y_0 = \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2) = -\frac{2a}{t_1}$ by (1)

$2x_0 = a(t_1^2 + t_2^2)$, $t_1 = -\frac{2a}{y_0}$ .... (2)

$= a[t_1^2 + \left(-t_1 - \frac{2}{t_1}\right)^2]$ by (1)

$= a\left[\left(-\frac{2a}{y_0}\right)^2 + \left(\frac{2a}{y_0} + \frac{y_0}{a}\right)^2\right]$

$\Rightarrow 2axy^2 = y^4 + 4a^2y^2 + 8a^4$



21. **<u>Concyclic points</u>**

In general, there are at most 4 concyclic points to a parabola.
The relation between the parameters are $t_1 + t_2 + t_3 + t_4 = 0$

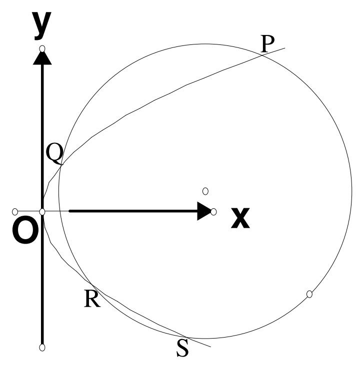
The parametric equation $\begin{cases} x = at^2 \\ y = 2at \end{cases}$ intersects with the circle.

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore (at^2)^2 + (2at)^2 + 2g(at^2) + 2f(2at) + c = 0$

$a^2t^4 + (4a^2 + 2ag)t^2 + (4af)t + c = 0$

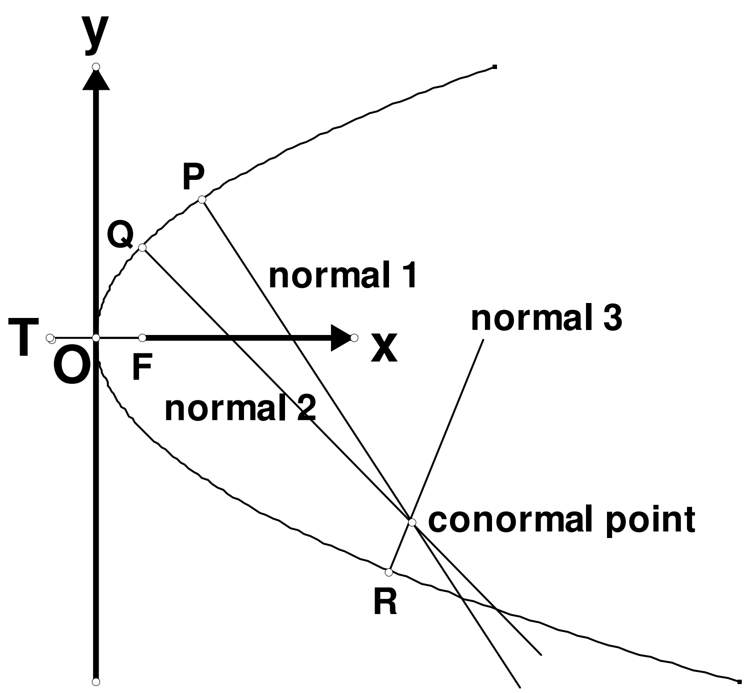
This is an equation in t of degree 4, which has at most 4 real roots t_1, t_2, t_3, t_4.



It means that the circle intersects the parabola in at most 4 points.

$t_1 + t_2 + t_3 + t_4 = \text{sum of roots} = -\frac{0}{a^2} = 0$, which is the required condition.

22. **<u>Conormal point</u>**



There are at most 3 points at which the normal drawn are concurrent at a point.

The relation between the parameter of the feet of normals are $t_1 + t_2 + t_3 = 0$

Equation of normal is $tx + y - 2at - at^3 = 0$

Since they meet at (h, k) (say)

$\therefore -at^3 + (h - 2a)t + k = 0$

This is an equation in t of degree 3, which has at most 3 real roots in t.

Therefore, there are at most 3 normals concurrent at (h, k)

$t_1 + t_2 + t_3$ = sum of roots $= -\dfrac{0}{-a} = 0$, which is the required condition.