4. The coordinates of points P and Q are (35, –50) and (59, –32).
A circle C passes through points P, Q and R.
The tangent of C at P cuts the straight line which passes through Q and R at S.
It is found that the coordinates of S are (65, –40).
(a) (i) Find the coordinates of R.
(ii) Find the equation of C.
(b) T is a point in quadrant I and lying on C such that the area of ΔTPS is half of the area of ΔRPS.
(i) Find the coordinates of T.
(ii) Someone claimns that the orthocentre of ΔTPS lies inside ΔTPS.
Do you agree? Explain your answer.
(c) Denote the centroid of ΔTPS by I.
(i) Describe the geometrical relationship among points I, Q and S.
(ii) Find the ratio of area of ΔRPS to that of ΔQPS.

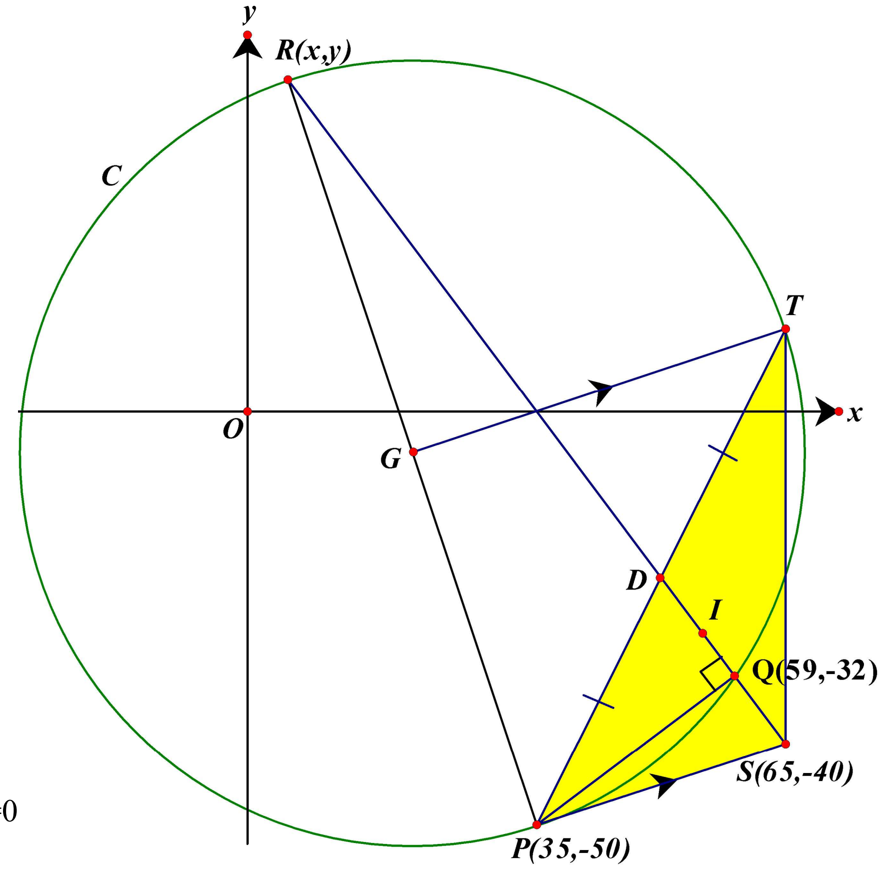
(a) (i) Denote m as the symbol for slopes.

$$m_{PQ}=\frac{-32-(-50)}{59-35}=\frac{18}{24}=\frac{3}{4}$$

$$m_{QS}=\frac{-32-(-40)}{59-65}=\frac{8}{-6}=-\frac{4}{3}$$

$$m_{PQ}\times m_{QS}=-1$$

$\therefore PQ\perp QS$ …… (*)



PR is the diameter of the circle
$\therefore RP\perp PS$ (tangent $\perp$ radius)
Equation of PS:

$$\frac{y-(-50)}{x-35}\times\frac{-40-(-50)}{65-35}=-1$$

$3x + y - 55 = 0$ …… (1)

Equation of QS:

$$\frac{y-(-40)}{x-65}=\frac{-32-(-40)}{59-65}$$

$4x + 3y - 140 = 0$ …… (2)

3(1) – (2): $5x - 25 = 0$
$x = 5$
Sub. $x = 5$ into (1): $15 + y - 55=0$
$y = 40$
The coordinates of R are (5, 40).

(ii) Let the centre be G. G is the mid-point of PR.

$$G=\left(\frac{5+35}{2},\frac{40+(-50)}{2}\right)=(20,-5)$$

$$\text{Radius} = GR=\sqrt{(20-5)^2+(-5-40)^2}=\sqrt{2250}$$

Equation of C: $(x - 20)^2 + (y + 5)^2 = 2250$ …… (3) (equivalently $x^2 + y^2 - 40x + 10y - 1825 = 0$)

(b) (i) Through G, draw GT // PS, cutting the circle C at T in the first quadrant.
ΔRPS, ΔGPS and ΔTPS have the same base (PS) but different heights.
$\because$ G (centre of circle) is the mid-point of RP and $\angle GPS = 90°$ (proved in (a)(i) (*))
Area of ΔGPS = half of the area of ΔRPS = area of ΔTPS
$\therefore$ Height of ΔTPS = height of ΔGPS = GP

$$\text{Equation of } GT:\ y-(-5)=\frac{-40-(-50)}{65-35}(x-20)$$

$3y + 15 = x - 20 \Rightarrow x = 3y + 35$ …… (4)

Sub. (4) into (3): $(3y + 35 - 20)^2 + (y + 5)^2 = 2250$
$9(y + 5)^2 + (y + 5)^2 = 2250$
$(y + 5)^2 = 225 = 15^2$
$y = 10$ or -20 (rejected)
Sub. $y = 10$ into (4): $x = 30 + 35 = 65$
The coordinates of T are (65, 10).

(ii) $PS = \sqrt{(65-35)^2 + [-40-(-50)]^2} = \sqrt{1000}$

$TP = \sqrt{(65-35)^2 + [10-(-50)]^2} = \sqrt{4500}$

$TS = 10 - (-40) = 50$ ($\because$ x-coordinate of T = x-coordinate of S)

$\cos \angle PST = \dfrac{PS^2 + ST^2 - PT^2}{2PS \cdot ST} = \dfrac{1000 + 2500 - 4500}{2\sqrt{1000} \cdot 50} < 0$

$\therefore\ \angle PST > 90°$
The orthocentre of ΔTPS lies outside ΔTPS
The claim is disagreed.

(c) (i) Let D be the mid-point of TP. $D = \left(\dfrac{35+65}{2}, \dfrac{10+(-50)}{2}\right) = (50, -20)$

The incentre of ΔTPS divides SD in the ratio 2 : 1.

$I = \left(\dfrac{50 \times 2 + 65}{1+2}, \dfrac{-20 \times 2 + (-40)}{1+2}\right) = \left(55, -\dfrac{80}{3}\right)$

$m_{IS} = \dfrac{-\frac{80}{3} - (-40)}{55 - 65} = -\dfrac{4}{3}$

$m_{QS} = \dfrac{-32 - (-40)}{59 - 65} = -\dfrac{4}{3}$

$\because\ m_{IS} = m_{QS}$

$\therefore$ I¸ Q and S are collinear

(ii) Denite S as the symbol for areas

$S_{\Delta RPS} : S_{\Delta QPS} = \dfrac{1}{2} RS \cdot PQ : \dfrac{1}{2} QS \cdot PQ$
$= RS : QS$
$= \sqrt{(65-5)^2 + [-40-(40)]^2} : \sqrt{(65-59)^2 + [-40-(-32)]^2}$
$= 100 : 10$
$= 10 : 1$