Family of circles

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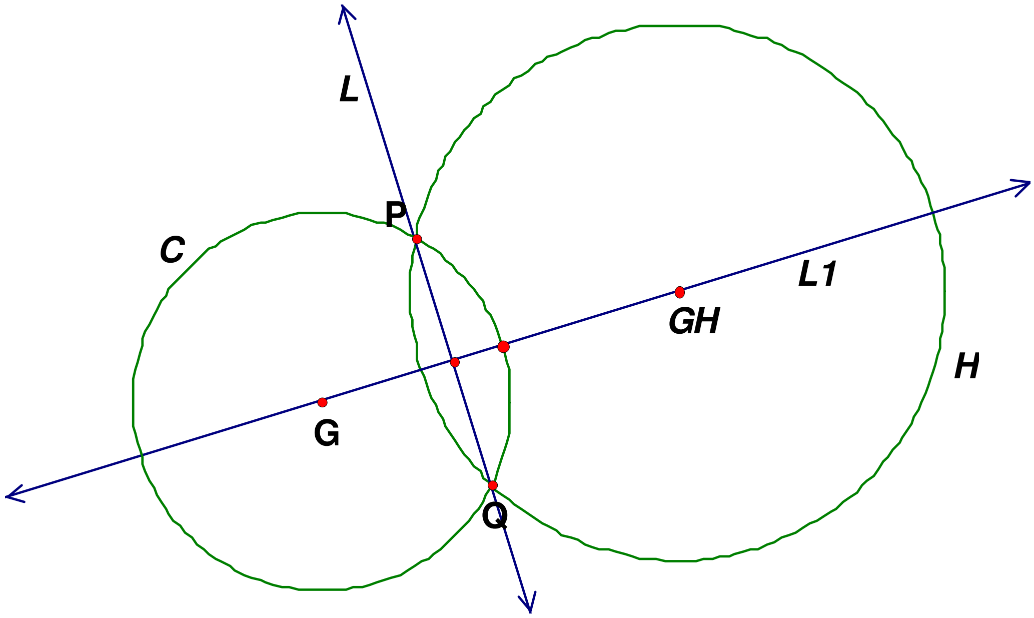
C: $x^2 + y^2 + Dx + Ey + F = 0$

L: $Ax + By + C = 0$

H: $C + kL$: $x^2 + y^2 + (D + kA)x + (E + kB)y + F + kC = 0$

If C and L intersect, the centres of the family of circles H passes through the perpendicular bisector of C and L.



centre of $H = G_H = \left(-\frac{D+kA}{2}, -\frac{E+kB}{2}\right)$

We shall find the locus of G_H by eliminating k.

Let $x = -\frac{D+kA}{2}$, $y = -\frac{E+kB}{2}$

$-2x - D = kA$; $-2y - E = kB$

$-\frac{2x+D}{A} = -\frac{2y+E}{B}$

$2Bx - 2Ay + BD - AE = 0$

This is the locus of centre of H, which is a straight line (let it be L_1)

slope of $L \times$ slope of $L_1 = -\frac{A}{B} \cdot \frac{B}{A} = -1$

$\therefore L \perp L_1$

Next, we shall show that the centre of C lies on L_1.

Sub. the centre $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ into L_1

LHS $= 2B\left(-\frac{D}{2}\right) - 2A\left(-\frac{E}{2}\right) + BD - AE$

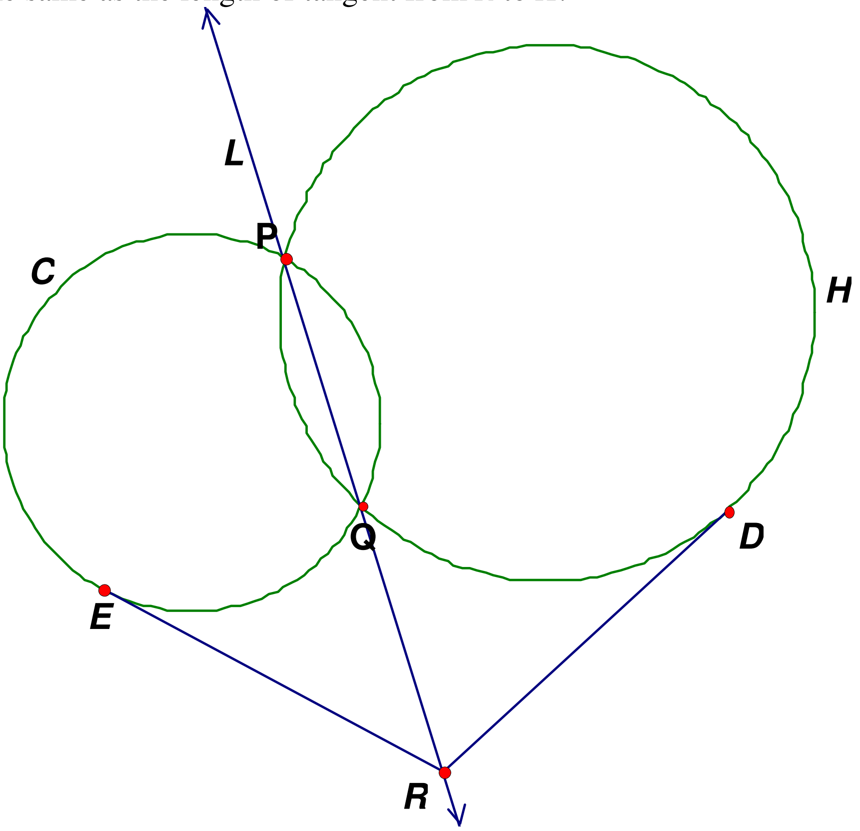
$= -BD + AE + BD - AE = 0 =$ RHS

$\therefore L_1$ passes through the centre of C. ($\perp$ from centre bisects chord)

So, L_1 is the perpendicular bisector of the intersection of C and L.

Note: If C and L does not intersect, the centre of H lie on a line L_1 which is perpendicular to L and passes through the centre of C.

If $R(x_0, y_0)$ is any point on L outside both circles C and H, then the length of tangent from R to C is the same as the length of tangent from R to H.



Suppose tangents RE touches C at E, RD touches H at D.

$RE^2 = x_0^2 + y_0^2 + Dx_0 + Ey_0 + F$

$RD^2 = x_0^2 + y_0^2 + (D + kA)x_0 + (E + kB)y_0 + F + kC$

$\quad = x_0^2 + y_0^2 + Dx_0 + Ey_0 + F + k(Ax_0 + By_0 + C)$

$\quad = x_0^2 + y_0^2 + Dx_0 + Ey_0 + F + k(0)$, $\because$ $R(x_0, y_0)$ lies on L

$\quad = RE^2$

$\therefore RD = RE$

Note: The result is also true even if C and L does not intersect.

L is called the **radical axis** of C and H.

Given the following two different circles:
C_1: $x^2 + y^2 + D_1x + E_1y + F_1 = 0$
C_2: $x^2 + y^2 + D_2x + E_2y + F_2 = 0$
$mC_1 + nC_2$: $m(x^2 + y^2 + D_1x + E_1y + F_1) + n(x^2 + y^2 + D_2x + E_2y + F_2) = 0$
$(m + n)x^2 + (m + n)y^2 + (mD_1 + nD_2)x + (mE_1 + nE_2)y + mF_1 + nF_2 = 0$ ....(*)
It represents a family of circles.

$C_1 - C_2$: $(D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2 = 0$ is called the **radical axis**. (called it L)

(1) $mC_1 + nC_2$ is the same as $C_1 + kL$.

Proof: Divide (*) by $(m + n)$

$$x^2 + y^2 + \frac{mD_1 + nD_2}{m+n}x + \frac{mE_1 + nE_2}{m+n}y + \frac{mF_1 + nF_2}{m+n} = 0$$

$$x^2 + y^2 + \frac{mD_1 + nD_1 + n(D_2 - D_1)}{m+n}x + \frac{mE_1 + nE_1 + n(E_2 - E_1)}{m+n}y + \frac{mF_1 + nF_1 + n(F_2 - F_1)}{m+n} = 0$$

$$x^2 + y^2 + D_1x + E_1y + F_1 + \frac{n(D_2 - D_1)}{m+n}x + \frac{n(E_2 - E_1)}{m+n}y + \frac{n(F_2 - F_1)}{m+n} = 0$$

$$x^2 + y^2 + D_1x + E_1y + F_1 + \frac{n}{m+n}[(D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2] = 0$$

which is the form $C_1 + kL$, where $k = \frac{n}{m+n}$.

(2) The radical axis passes through a line which is perpendicular to the line joining the centres of C_1 and C_2.

Proof: L: $(D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2 = 0$

Centres $G_1\left(-\frac{D_1}{2}, -\frac{E_1}{2}\right)$, $G_2\left(-\frac{D_2}{2}, -\frac{E_2}{2}\right)$

$$\text{Product of slopes} = -\frac{D_1 - D_2}{E_1 - E_2} \cdot \frac{-\frac{E_2}{2} + \frac{E_1}{2}}{-\frac{D_2}{2} + \frac{D_1}{2}}$$

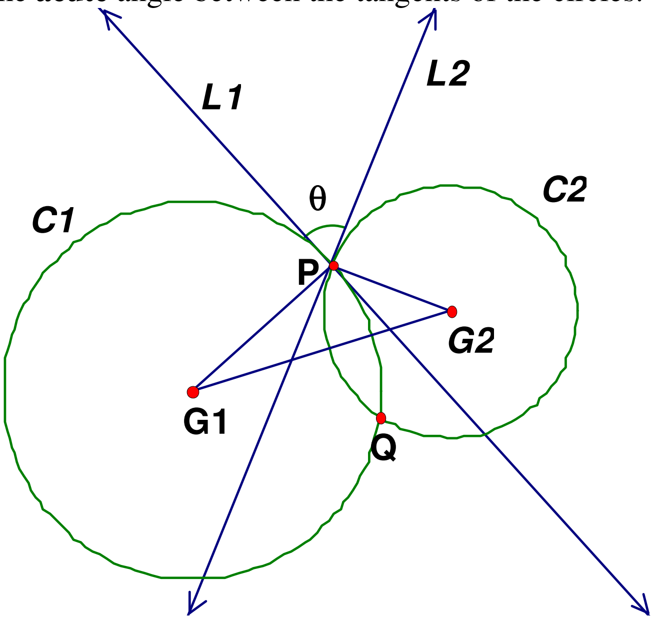
$$= -\frac{D_1 - D_2}{E_1 - E_2} \cdot \frac{E_1 - E_2}{D_1 - D_2} = -1$$

$\therefore$ They are perpendicular.

(3) If C_1 and C_2 intersect at P, Q, then PQ is the radical axis.

(4) $C_1 + kL$ and C_1 have the same radical axis.

Given the following two different circles:
C_1: $x^2 + y^2 + D_1x + E_1y + F_1 = 0$
C_2: $x^2 + y^2 + D_2x + E_2y + F_2 = 0$
Suppose C_1 and C_2 intersect at P and Q.
Find the **acute** angle between the tangents of the circles.



Suppose the centres of C_1 and C_2 are G_1 and G_2 respectively.
Suppose the radii of C_1 and C_2 are r_1 and r_2 respectively.
Suppose the distance between the centres G_1 and G_2 is d.
Let L_1 be the tangent at P to the circle C_1, let L_2 be the tangent at P to the circle C_2.
Let θ be the angle between L_1 and L_2.
$G_1P \perp L_1$, $G_2P \perp L_2$ (tangent ⊥ radius)
Consider the sum of angles at P: $\theta + 90° + 90° + \angle G_1PG_2 = 360°$ (∠s at a point)
$\theta = 180° - \angle G_1PG_2$
By cosine formula on ΔG_1PG_2

$$\cos \angle G_1PG_2 = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

$$\cos \theta = -\cos \angle G_1PG_2 = \frac{d^2 - (r_1^2 + r_2^2)}{2r_1r_2}$$

Note that θ may be acute or obtuse depend on $\cos \theta > 0$ or $\cos \theta < 0$.

Orthogonal circles
Two circles are orthogonal if the angle between them is 90°
In this case, $r_1^2 + r_2^2 = G_1G_2^2$.

Exercise
Show that the circles
C_1: $x^2 + y^2 - 6x = 0$
C_2: $x^2 + y^2 - 8y = 0$
are orthogonal.