Lecture Notes on Coordinate Geometry: Ellipse

Reference: Advance Level Pure Mathematics by S.L. Green p.72-87 2-5-2006 Last updated: 30/08/2021

1. Definition

Given a fixed point S and a straight line DD' (called the **directrix**). An **ellipse** is the locus of a variable point P for which the ratio of distance SP to the distance from P to DD' is always equal to a constant e, where $0 < e < 1$. (e is the eccentricity)

Let M and N be the feet of perpendiculars drawn from P and S onto the directrix respectively.

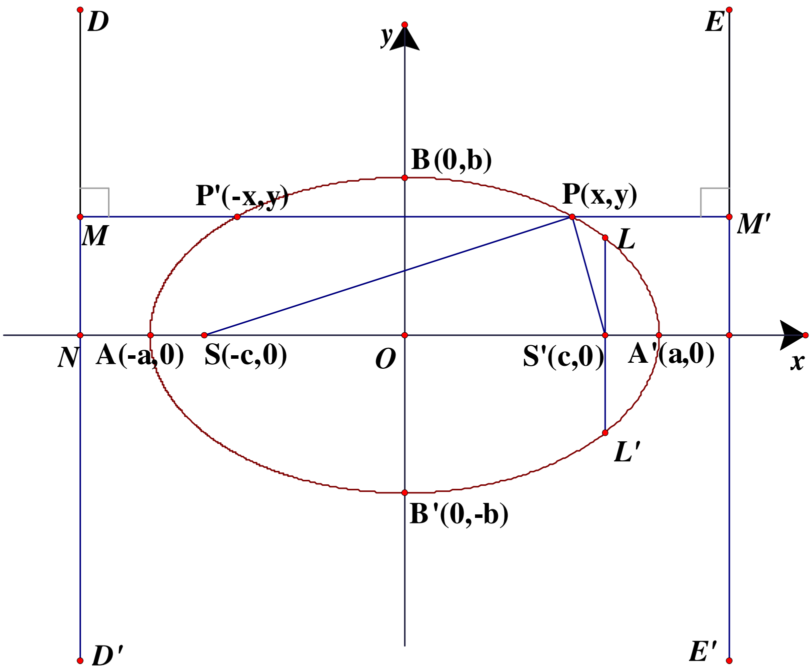
Then by definition: $\dfrac{SP}{PM} = e < 1$.

In particular, when P moves to A between S and N, $AS = eAN$

Produce AS further to a point A' such that $A'S = eA'N$; then A' is on the curve.

Bisect AA' at $O(0, 0)$ (called the **centre**) and let $AA' = 2a$, then $A = (a, 0)$, $A' = (-a, 0)$.

Let DD' be $x = -d$, then $N(-d, 0)$.



$$\frac{SA}{AN} = e = \frac{SA'}{A'N}$$

$SA = a - c$, $AN = d - a$, $SA' = a + c$, $A'N = a + d$

$$\frac{a-c}{d-a} = \frac{a+c}{d+a} = e \cdots\cdots (1)$$

Cross multiplying: $a^2 - ac + ad - cd = ad - a^2 + cd - ac$

$a^2 = cd$

$$\Rightarrow d = \frac{a^2}{c} \quad \cdots\cdots (2)$$

Sub. (2) into (1): $\dfrac{a+c}{a+\dfrac{a^2}{c}}=e$

$\dfrac{c(a+c)}{a(a+c)}=e$

$\therefore c = ae$ …… (3)

Sub. (3) into (2): $d=\dfrac{a^2}{c}=\dfrac{a^2}{ae}=\dfrac{a}{e}$

$\therefore d=\dfrac{a}{e}$ …… (4)

Let $P = (x, y)$, then $SP=\sqrt{(x+c)^2+y^2}$, $PM = x + d = x+\dfrac{a}{e}$

$\because \dfrac{SP}{PM}=e \Rightarrow \dfrac{\sqrt{(x+c)^2+y^2}}{x+\dfrac{a}{e}}=e$

$(x + ae)^2 + y^2 = (ex + a)^2$

$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$

$(1 - e^2)x^2 + y^2 = a^2(1 - e^2)$

$\left(\dfrac{x}{a}\right)^2+\dfrac{y^2}{a^2(1-e^2)}=1$

$\because 0 < e < 1$, let $b^2 = a^2(1 - e^2) \Rightarrow a^2 - b^2 = a^2e^2 > 0$

$\Rightarrow a^2 - b^2 = c^2$ …… (5)

The equation of an ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ …… (6)

2. Put $x = 0$ into (6): $y = \pm b$, The points $A(-a, 0)$, $A'(a, 0)$, $B(0, b)$, $B'(0, -b)$ are the **vertices** of the ellipse.

From (6): $\left(\frac{x}{a}\right)^2 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow -1 \le \frac{x}{a} \le 1 \Rightarrow -a \le x \le a$

Also: $\left(\frac{y}{b}\right)^2 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow -1 \le \frac{y}{b} \le 1 \Rightarrow -b \le y \le b$

$\therefore$ There is no point for $x > a$, $x < -a$, $y > b$ and $y < -b$.

Replace x by $-x$ in (6): $\frac{(-x)^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

There is no change. $\therefore$ **The curve is symmetric about *y*-axis.**

Replace y by $-y$ in (6): $\frac{x^2}{a^2} + \frac{(-y)^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

There is no change. $\therefore$ **The curve is symmetric about *x*-axis.**

For any point $P(x, y)$ lies on the ellipse, $P'(-x, y)$ is the image of P, also lies on the ellipse.

($\because$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetric.)

Let $S'(c, 0)$ be the image of S reflected along y-axis. $x = d$ is the image of the directrix $x = -d$.

Then $\frac{S'P'}{P'M'} = e$ (all dashes are images.)

P is the image of P', which lies on the curve.

$\therefore \frac{S'P}{PM'} = e$ for any point P on the ellipse.

$\therefore$ There are **two foci** $S(-c, 0)$, $S'(c, 0)$ and **two directrices** $x = -d$, $x = d$.

$AA' = 2a$ is the **major axis**, $BB' = 2b$ is the **minor axis**, a = **semi-major axis**, b = **semi-minor axis**.

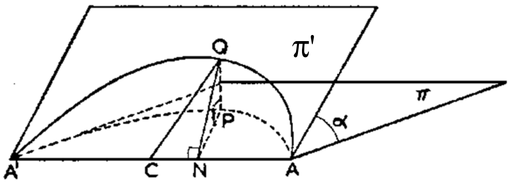
They are the **principal axes**.

3. The **latus rectum** LL' is a line segment through the focus $S'(c, 0)$ perpendicular to the x-axis cutting the ellipse at L and L'

Put $x = c$ into (6): $\frac{c^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm b\sqrt{1 - \frac{c^2}{a^2}} = \pm b\sqrt{\frac{a^2 - c^2}{a^2}} = \pm\frac{b^2}{a}$ (by (5) $a^2 - c^2 = b^2$)

$\therefore L = (c, \frac{b^2}{a})$, $L' = (c, -\frac{b^2}{a}) \Rightarrow LL' = 2\frac{b^2}{a}$ $\cdots\cdots$ (7)

4. An ellipse as the **orthogonal projection** of a **circle**.



Let π and π' be two planes inclined at an angle α and intersect at *A'CNA*.

A circle is drawn on π' with *C* as centre, *CA* = *CA'* = radius passing through *AQA* = *a*.

If *N* is the foot of perpendicular drawn from *Q* onto *AA'*. *P* is the projection of *Q* onto the plane π.

Using Pythagoras theorem on ΔCNQ, $CN^2 + NQ^2 = CQ^2 = a^2$ …… (8)

$NP = NQ \cos\alpha$, where α = angle of projection.

Let $\cos\alpha = \frac{b}{a}$, $NP = \frac{b}{a}NQ \Rightarrow NQ = \frac{a}{b}NP$ …… (9)

$\therefore$ Sub. (9) into (8): $CN^2 + \frac{a^2}{b^2}NP^2 = a^2$

$\therefore \frac{CN^2}{a^2} + \frac{NP^2}{b^2} = 1$, which is the equation of an ellipse.

$\therefore$ An ellipse may be regarded as the projection of a circle with the angle of projection = α.

If $(x, y) = (a\cos\theta, a\sin\theta)$ is the parametric equation of the circle on π', then

$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$ …… (10) is the **parametric equation** of the ellipse, where θ = eccentric angle.

$\left(y = NP = \frac{b}{a}NQ = \frac{b}{a}\cdot a\sin\theta = b\sin\theta\right)$

Example 1

Let E: $\frac{x^2}{25}+\frac{y^2}{3}=1$

L: $y = 25x + m$

Suppose there are two points P, Q on E which are symmetric about L.

What can you say about m?

Let $P = (5\cos\alpha, \sqrt{3}\sin\alpha)$, $Q = (5\cos\beta, \sqrt{3}\sin\beta)$

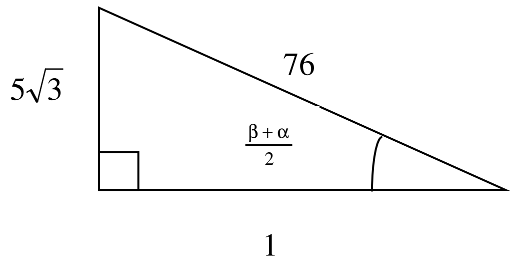
L is the perpendicular of PQ. $PQ \perp L$ and P, Q are equal distance to L.

Let M = mid-point of PQ = $\left(5\frac{\cos\alpha+\cos\beta}{2}, \sqrt{3}\frac{\sin\alpha+\sin\beta}{2}\right)$

M lies on L: $\frac{\sqrt{3}(\sin\alpha+\sin\beta)}{2}=\frac{125(\cos\alpha+\cos\beta)}{2}+m \cdots\cdots (11)$

$PQ \perp L$: $\frac{\sqrt{3}(\sin\beta-\sin\alpha)}{5(\cos\beta-\cos\alpha)}\times 25=-1 \quad \cdots\cdots (12)$

From (2): $\frac{5\sqrt{3}\left(2\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right)}{\left(-2\sin\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right)}=-1$



$\tan\frac{\beta+\alpha}{2}=5\sqrt{3} \quad \cdots\cdots (13)$

$\sin\frac{\beta+\alpha}{2}=\frac{5\sqrt{3}}{76} \quad \cdots\cdots (14)$

$\cos\frac{\beta+\alpha}{2}=\frac{1}{76} \quad \cdots\cdots (15)$

From (10): $\sqrt{3}\left(\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)=125\left(\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)+m \quad \cdots\cdots (16)$

Sub. (14) and (15) into (16)

$$\frac{15}{76}\left(\cos\frac{\alpha-\beta}{2}\right)=\frac{125}{76}\left(\cos\frac{\alpha-\beta}{2}\right)+m$$

$$-\frac{110}{76}\left(\cos\frac{\alpha-\beta}{2}\right)=m$$

$$\cos\frac{\alpha-\beta}{2}=-\frac{38m}{55}$$

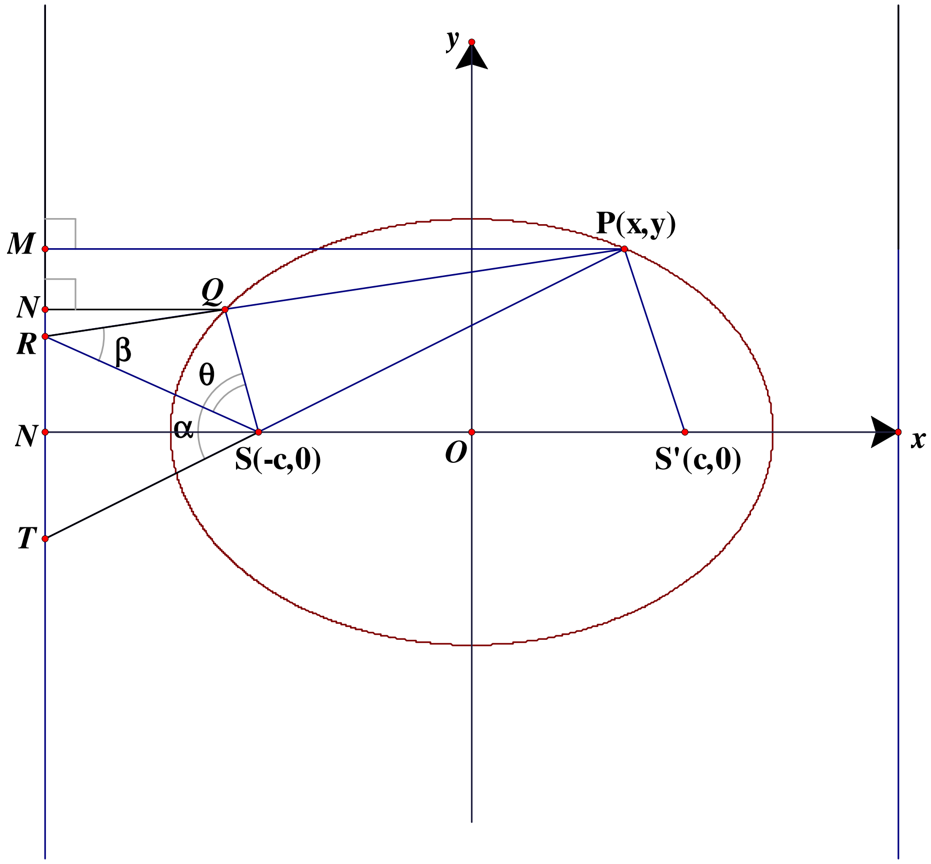
$$-1\le\cos\frac{\alpha-\beta}{2}\le 1$$

$$-1\le-\frac{38m}{55}\le 1$$

$$\frac{55}{38}\ge m\ge-\frac{55}{38}$$

5. Geometrical Property

If a chord PQ cuts a directrix at R, then $\boldsymbol{RS}$ **bisects the exterior angle of** $\boldsymbol{\angle PSQ}$.



Suppose the chord PQ produced intersects the directrix at R. PS produced intersects the directrix at T. Let M and N be the feet of perpendiculars drawn from P and Q respectively onto the directrix.

Let $\angle QRS = \beta$, $\angle QSR = \theta$, $\angle RST = \alpha$

By definition, $\dfrac{SP}{PM} = \dfrac{SQ}{QN} = e$

$\therefore \dfrac{SP}{SQ} = \dfrac{PM}{QN}$

But $\dfrac{PM}{QN} = \dfrac{PR}{QR}$ ($\because \Delta PMR \sim \Delta QNR$)

$\therefore \dfrac{SP}{SQ} = \dfrac{PR}{QR}$

$\dfrac{SP}{PR} = \dfrac{SQ}{QR}$ …… (17)

By sine law on ΔSPR, $\dfrac{SP}{PR} = \dfrac{\sin\beta}{\sin\alpha}$ …… (18)

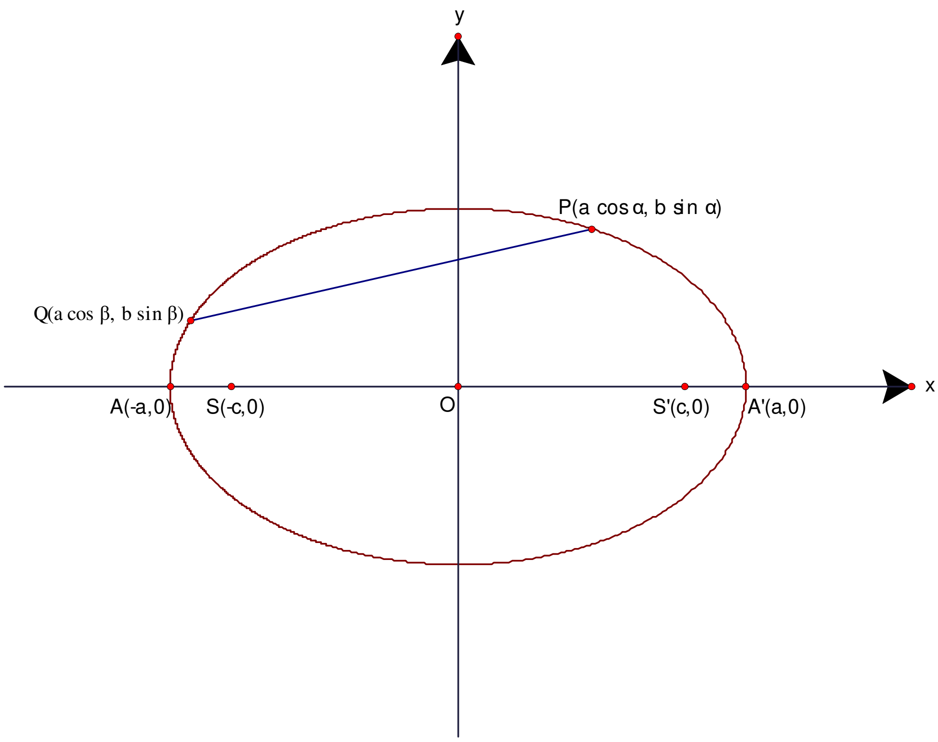
By sine law on ΔSQR, $\dfrac{SQ}{QR} = \dfrac{\sin\beta}{\sin\theta}$ …… (19)

By (17), $\dfrac{SP}{PR} = \dfrac{SQ}{QR} \Rightarrow \dfrac{\sin\beta}{\sin\alpha} = \dfrac{\sin\beta}{\sin\theta} \Rightarrow \alpha = \theta$ ($\because$ In the figure, $\alpha + \theta < 180°$, it is impossible that $\alpha = 180° - \theta$)

6. Equation of chord using parameters

Let α and β be the eccentric angles of two distinct points P and Q.

Then $P = (a \cos \alpha, b \sin \alpha)$, $Q = (a \cos \beta, b \sin \beta)$



Equation of PQ: $\dfrac{y-b\sin\beta}{x-a\cos\beta}=\dfrac{b\sin\alpha-b\sin\beta}{a\cos\alpha-a\cos\beta}$

$$\frac{y-b\sin\beta}{x-a\cos\beta}=\frac{b}{a}\cdot\frac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta}=\frac{b}{a}\cdot\frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\frac{y-b\sin\beta}{x-a\cos\beta}=-\frac{b\cos\frac{\alpha+\beta}{2}}{a\sin\frac{\alpha+\beta}{2}}$$

$$-ay\sin\frac{\alpha+\beta}{2}+ab\sin\beta\sin\frac{\alpha+\beta}{2}=bx\cos\frac{\alpha+\beta}{2}-ab\cos\beta\cos\frac{\alpha+\beta}{2}$$

$$(b\cos\frac{\alpha+\beta}{2})x+(a\sin\frac{\alpha+\beta}{2})y=ab(\cos\beta\cos\frac{\alpha+\beta}{2}+\sin\beta\sin\frac{\alpha+\beta}{2})$$

$$(b\cos\frac{\alpha+\beta}{2})x+(a\sin\frac{\alpha+\beta}{2})y=ab\cos(\beta-\frac{\alpha+\beta}{2})$$

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\beta-\alpha}{2}\quad\cdots\cdots(20)$$

7. Equation of tangent at θ

As $\beta \to \alpha = \theta$, the equation of chord becomes: $b \cos \theta\, x + a \sin \theta\, y = ab$

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ …… (21)

This is the equation of tangent with parameter θ.

If (x_0, y_0) lies on the ellipse (6): $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $x_0 = a \cos \theta$, $y_0 = b \sin \theta$

$\therefore$Equation of tangent: $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$ …… (22)

This is the equation of tangent passes through (x_0, y_0) on the ellipse.

8. Given a line of slope *m*, find the condition for tangency.

$y = mx + k$ is identical to (14): $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

$\therefore$The coefficients are in proportion: $\frac{x_0}{ma^2} = \frac{y_0}{-b^2} = \frac{-1}{k}$.

$\Rightarrow x_0 = -\frac{ma^2}{k}$, $y_0 = \frac{b^2}{k}$

$\because \left(\frac{x_0}{a}\right)^2 + \left(\frac{y_0}{b}\right)^2 = 1 \quad \therefore \left(-\frac{ma^2}{ak}\right)^2 + \left(\frac{b^2}{bk}\right)^2 = 1$

$\frac{a^2m^2}{k^2} + \frac{b^2}{k^2} = 1$

$k^2 = a^2m^2 + b^2$ …… (23)

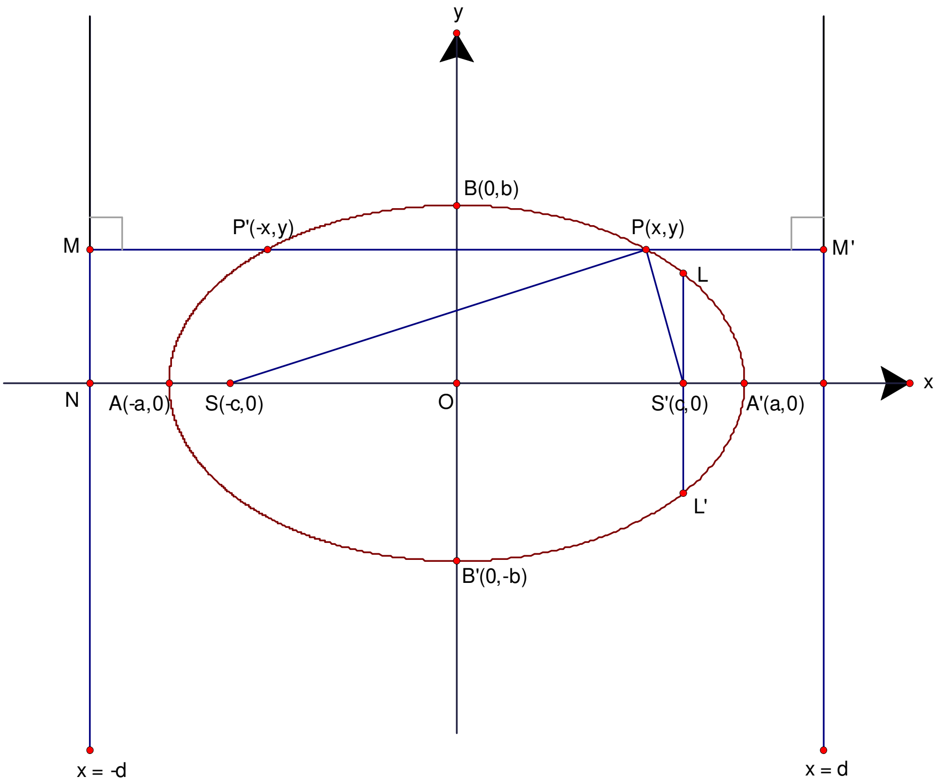
$k = \pm\sqrt{a^2m^2 + b^2}$

The equation of tangent given slope m is $y = mx \pm \sqrt{a^2m^2 + b^2}$ …… (24)

Method 2

Let $y = mx + k$ be the equation of a tangent.
Sub. into the ellipse: $b^2x^2 + a^2(mx + k)^2 = a^2b^2$
$(a^2m^2 + b^2)x^2 + 2a^2mkx + a^2(k^2 - b^2) = 0$
$\Delta = 4[(a^2mk)^2 - (a^2m^2 + b^2)a^2(k^2 - b^2)] = 0$
$a^2m^2k^2 - (a^2m^2k^2 + b^2k^2 - a^2b^2m^2 - b^4) = 0$
$b^2k^2 = a^2b^2m^2 + b^4$
$k^2 = a^2m^2 + b^2$
$k = \pm\sqrt{a^2m^2 + b^2}$
$\therefore$ Given slope m, the equation of tangent is $y = mx \pm \sqrt{a^2m^2 + b^2}$

9. The sum of distance of any point on the ellipse to the two foci is a constant (= 2*a*).



$SP + S'P = e(PM + PM') = e\ MM' = 2ed = 2a$

$\therefore\ SP + S'P = 2a \cdots\cdots (25)$

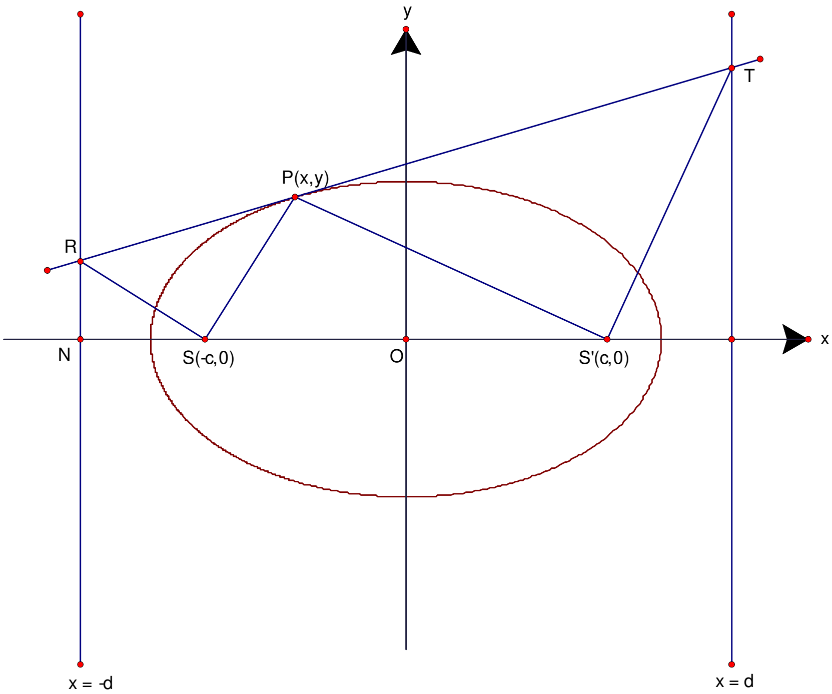
On the other hand, $SP = ePM$, $S'P = e\ PM'$

$P = (a \cos \theta, b \sin \theta)$

$PM = a \cos \theta + d = a \cos \theta + \frac{a}{e}$, $PM' = d - a \cos \theta = \frac{a}{e} - a \cos \theta$

$SP + S'P = e\ PM + e\ PM' = e(a \cos \theta + \frac{a}{e} + \frac{a}{e} - a \cos \theta) = 2a \cdots\cdots (26)$

10. If a tangent at P (on the ellipse) cuts the directrix ($x = -d$) at R, then $\angle PSR = 90°$



Proof: $S = (-c, 0) = (-ae, 0)$. The directrix $x = -d = -\frac{a}{e}$

Let m_1 = slope of $SP = \frac{y_0}{x_0 + ae}$, m_2 = slope of SR

R is giving by solving $\begin{cases} \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1 \\ x = -\frac{a}{e} \end{cases}$.

$$-\frac{x_0}{ae} + \frac{y_0 y}{b^2} = 1$$

$$y = \frac{b^2}{y_0}\left(1 + \frac{x_0}{ae}\right) = \frac{b^2(ae + x_0)}{aey_0}$$

$$\therefore R = \left(-\frac{a}{e}, \frac{b^2(ae + x_0)}{aey_0}\right)$$

$$m_2 = \frac{b^2(ae + x_0)}{aey_0\left(-\frac{a}{e} + ae\right)} = \frac{b^2(ae + x_0)}{a^2 y_0(e^2 - 1)} = -\frac{(ae + x_0)}{y_0} \quad (\because \text{By (5)}: a^2 - b^2 = c^2 = a^2e^2)$$

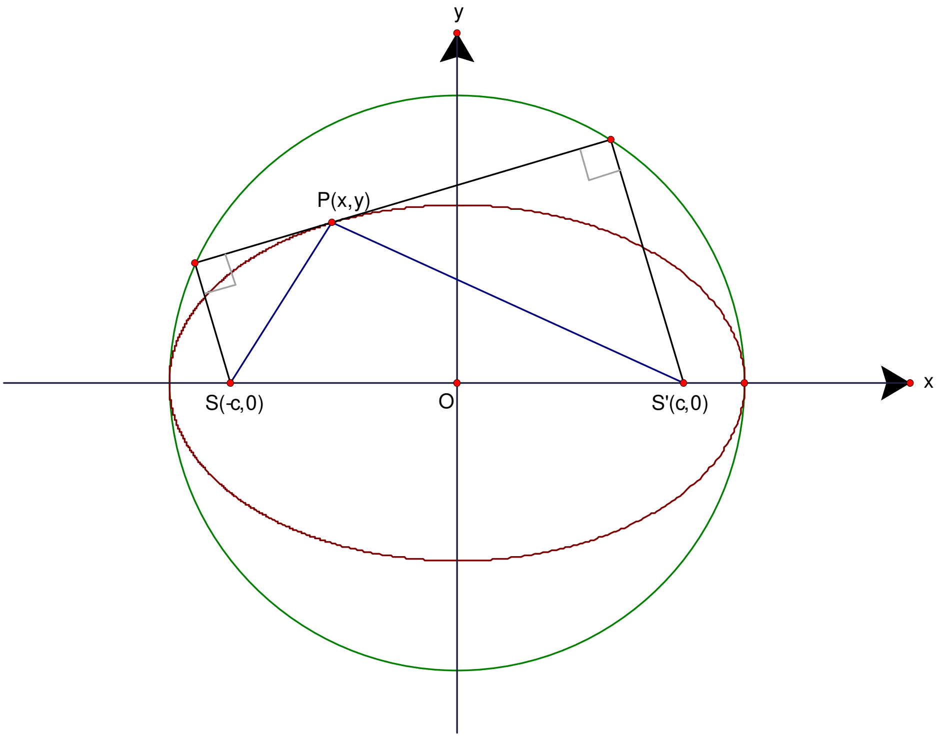
$$m_1 m_2 = \frac{y_0}{x_0 + ae} \times \left[-\frac{(ae + x_0)}{y_0}\right] = -1$$

$\therefore SP \perp SR$

Similarly, if the tangent at P (on the ellipse) cuts the directrix ($x = d$) at T, then $\angle PS'T = 90°$

11. The locus of foot of perpendiculars from a focus to a tangent is the auxiliary circle.

(Centre = $O(0, 0)$, radius = a)



From section 8, equation of tangent with a given slope is: $y - mx = \pm\sqrt{a^2m^2 + b^2}$ ⋯⋯ (24)

The equation of perpendicular line through $S'(ae, 0)$ is: $my + x = ae$ ⋯⋯ (27)

$(24)^2 + (27)^2$: $(1 + m^2)(x^2 + y^2) = a^2e^2 + a^2m^2 + b^2$

$= a^2e^2 + a^2m^2 + a^2(1 - e^2)$ (by (5))

$= a^2m^2 + a^2$

$= a^2(1 + m^2)$

$x^2 + y^2 = a^2$

Similarly, the equation of perpendicular line through $S(-ae, 0)$ is: $my + x = -ae$ ⋯⋯ (28)

$(24)^2 + (28)^2$: $(1 + m^2)(x^2 + y^2) = a^2e^2 + a^2m^2 + b^2$

$= a^2e^2 + a^2m^2 + a^2(1 - e^2)$ (by (5))

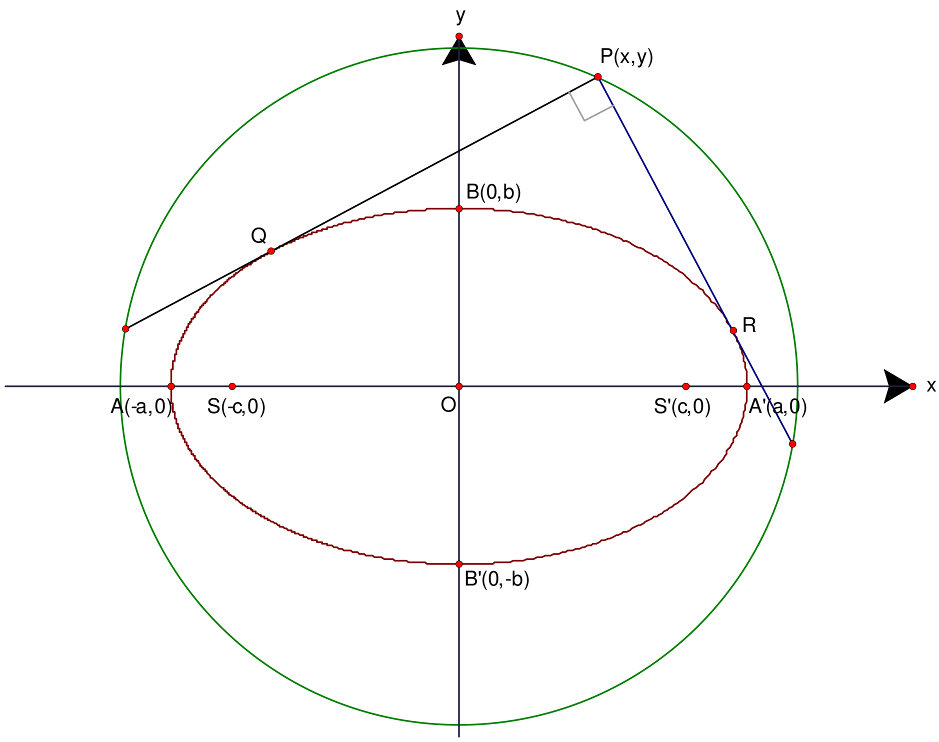
$= a^2m^2 + a^2$

$= a^2(1 + m^2)$

$x^2 + y^2 = a^2$

12. The locus of the intersection of the two perpendicular tangents is the director circle.

(Centre $O(0, 0)$, radius $= \sqrt{a^2 + b^2}$)



From section 8, equation of tangent with a given slope is: $y = mx \pm \sqrt{a^2 m^2 + b^2}$ $\cdots\cdots$ (24)

$(y - mx)^2 = m^2 a^2 + b^2$

$(a^2 - x^2)m^2 + 2xym + b^2 - y^2 = 0$

$\because$ The two tangents are perpendicular,

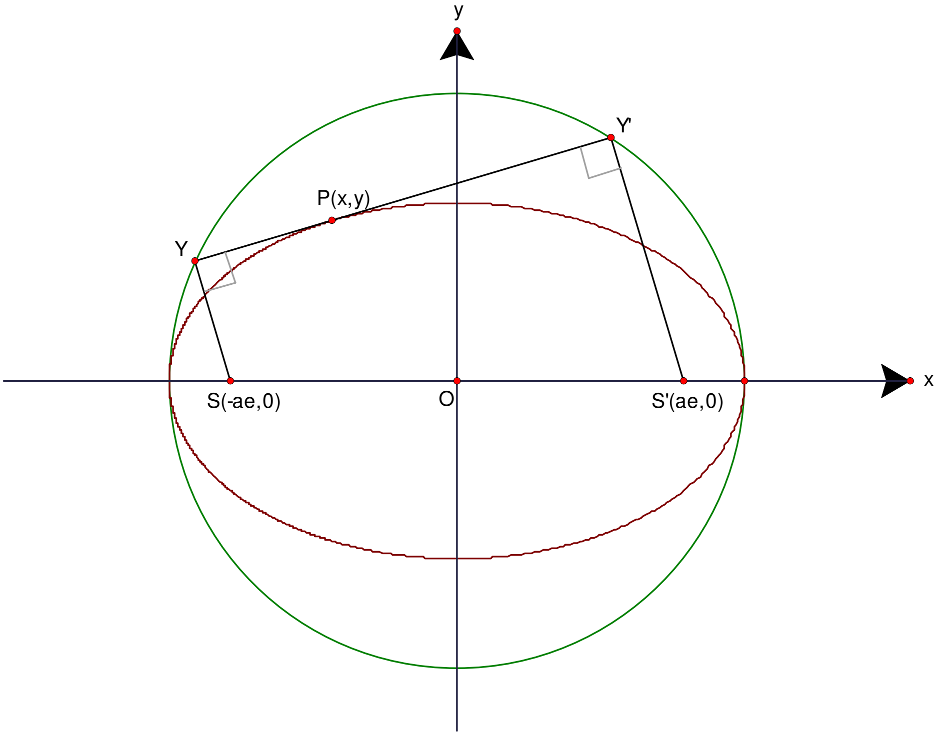
$\therefore$ $m_1 m_2$ = product of roots of the above quadratic equation in $m = -1$

$$\frac{b^2 - y^2}{a^2 - x^2} = -1$$

$x^2 + y^2 = a^2 + b^2$ $\cdots\cdots$ (29)

This is the equation of the **director circle**.

13. The product of two perpendiculars from the two foci to the tangent is a constant (= b^2)



Let the equation of the tangent be $y = mx \pm \sqrt{a^2m^2+b^2}$ …… (24). Let Y and Y' be the feet of perpendicular drawn form the two foci S and S' onto the tangent respectively.

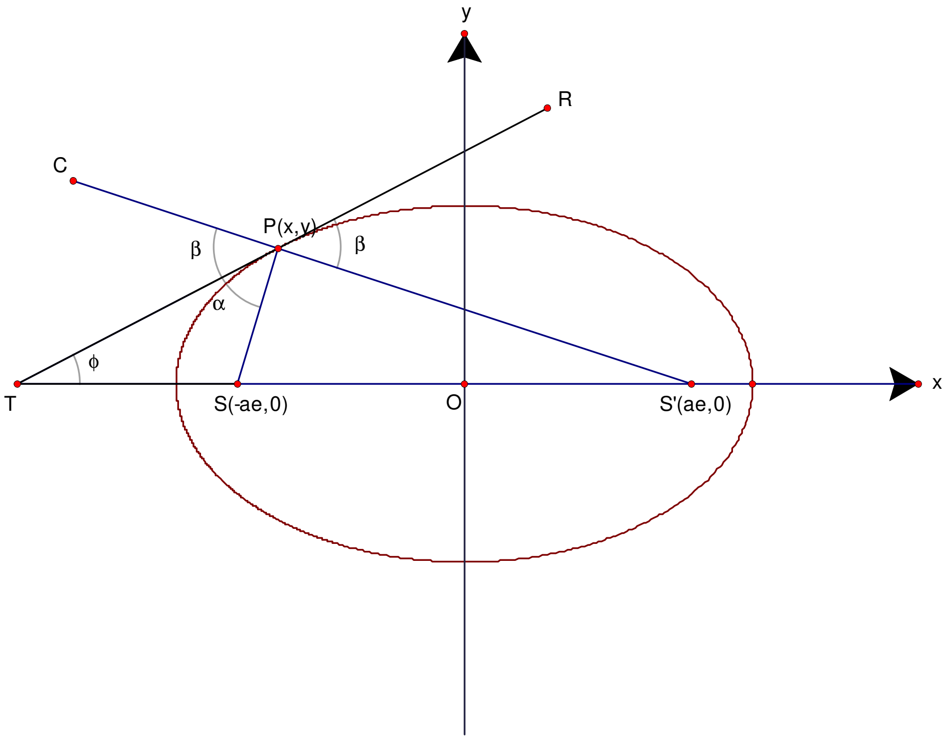
Then by distance formula, $SY \cdot S'Y' = \dfrac{m(ae) \pm \sqrt{m^2a^2+b^2}}{\sqrt{1+m^2}} \cdot \dfrac{m(-ae) \pm \sqrt{m^2a^2+b^2}}{\sqrt{1+m^2}}$

$$= \frac{m^2a^2+b^2-a^2e^2m^2}{1+m^2}$$

$$= \frac{m^2b^2+b^2}{1+m^2} \quad (\because \text{ by (5), } a^2 - a^2e^2 = b^2)$$

$$= b^2$$

14. The tangent makes equal angles with the focal distance to the point of contact.



In the figure, the tangent (RPT) at P on the ellipse cuts the x-axis at T. $\angle SPT = \alpha$, $\angle S'PR = \beta$. To prove that $\alpha = \beta$. (If the tangent at P does not cut x-axis, then $P = B(0, b)$ or $B'(0, -b)$, clearly $\alpha = \beta$)

Produce $S'P$ to C. Then $\angle CPT = \beta$ (vert. opp. $\angle$s)

$P = (a \cos \theta, b \sin \theta)$

$$SP = \sqrt{(a\cos\theta + ae)^2 + (b\sin\theta)^2} = \sqrt{a^2\cos^2\theta + 2a^2 e\cos\theta + a^2e^2 + b^2\sin^2\theta}$$

$$= \sqrt{a^2\cos^2\theta + 2a^2 e\cos\theta + a^2e^2 + (a^2 - c^2)\sin^2\theta}, \text{ (by (5), } b^2 = a^2 - c^2\text{)}$$

$$= \sqrt{a^2 + 2ac\cos\theta + c^2 - c^2\sin^2\theta}, \ (\because ae = c)$$

$$= \sqrt{a^2 + 2ac\cos\theta + c^2\cos^2\theta}$$

$$= \sqrt{(a + c\cos\theta)^2} = a + c\cos\theta = a + ae\cos\theta \text{ (see equation (26))}$$

$$S'P = \sqrt{(a\cos\theta - ae)^2 + (b\sin\theta)^2} = \sqrt{a^2\cos^2\theta - 2a^2 e\cos\theta + a^2e^2 + b^2\sin^2\theta}$$

$$= \sqrt{a^2\cos^2\theta - 2a^2 e\cos\theta + a^2e^2 + (a^2 - c^2)\sin^2\theta}, \text{ (by (5), } b^2 = a^2 - c^2\text{)}$$

$$= \sqrt{a^2 - 2ac\cos\theta + c^2 - c^2\sin^2\theta}, \ (\because ae = c)$$

$$= \sqrt{a^2 - 2ac\cos\theta + c^2\cos^2\theta}$$

$$= \sqrt{(a - c\cos\theta)^2} = a - c\cos\theta = a - ae\cos\theta \text{ (see equation (26))}$$

$SS' = 2c = 2ae$

Equation of PT: $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ …… (21)

To find T: let $y = 0$, $x = a\sec\theta$

$ST = -ae - a\sec\theta$, $S'T = ae - a\sec\theta$

Apply sine formula on ΔPST: $\frac{SP}{\sin\phi} = \frac{ST}{\sin\alpha} \Rightarrow \frac{a + ae\cos\theta}{\sin\phi} = \frac{-ae - a\sec\theta}{\sin\alpha}$ …… (30)

Apply sine formula on $\Delta PS'T$: $\frac{S'P}{\sin\phi} = \frac{S'T}{\sin(180^\circ - \beta)} \Rightarrow \frac{a - ae\cos\theta}{\sin\phi} = \frac{ae - a\sec\theta}{\sin\beta}$ …… (31)

(30)÷(31):

$$\frac{a + ae\cos\theta}{a - ae\cos\theta} = \frac{-ae - a\sec\theta}{ae - a\sec\theta} \cdot \frac{\sin\beta}{\sin\alpha}$$

$$\frac{a + ae\cos\theta}{a - ae\cos\theta} = \frac{ae + \frac{a}{\cos\theta}}{-ae + \frac{a}{\cos\theta}} \cdot \frac{\sin\beta}{\sin\alpha}$$

$$\frac{a + ae\cos\theta}{a - ae\cos\theta} = \frac{ae\cos\theta + a}{-ae\cos\theta + a} \cdot \frac{\sin\beta}{\sin\alpha}$$

$\sin\alpha = \sin\beta$

$\alpha = \beta$ or $\alpha = 180^\circ - \beta$

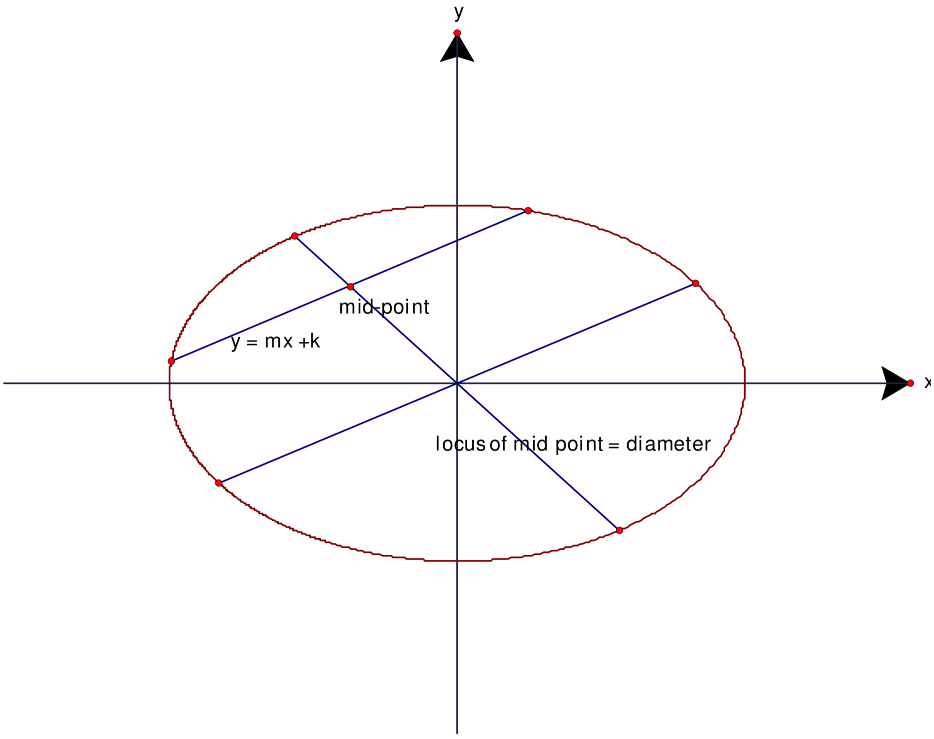
But $\alpha + \beta < 180^\circ$ (in the figure), $\therefore$ rejected

$\therefore \alpha = \beta$

***PT* is the exterior bisector of $\Delta PSS'$.**

Equivalently, the angle bisector of $\angle SPS'$ is the normal at *P*.

15. Find the locus of mid points of parallel chords.



Given the slope = m, the equation of chord is $y = mx + k$.

The points of intersection is given by $\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \\ y=mx+k \end{cases}$

Substitute: $\dfrac{x^2}{a^2}+\dfrac{(mx+k)^2}{b^2}=1 \Rightarrow \left(\dfrac{1}{a^2}+\dfrac{m^2}{b^2}\right)x^2+\dfrac{2kmx}{b^2}+\dfrac{k^2}{b^2}-1=0$

If (p, q) is the mid point of chord, then $p=\dfrac{x_1+x_2}{2}=-\dfrac{km}{b^2}\cdot\dfrac{a^2b^2}{b^2+a^2m^2}=-\dfrac{a^2km}{b^2+a^2m^2}$

$q = mp + k = -\dfrac{a^2km^2}{b^2+a^2m^2}+k=\dfrac{-a^2km^2+b^2k+a^2km^2}{b^2+a^2m^2}=\dfrac{b^2k}{b^2+a^2m^2}$

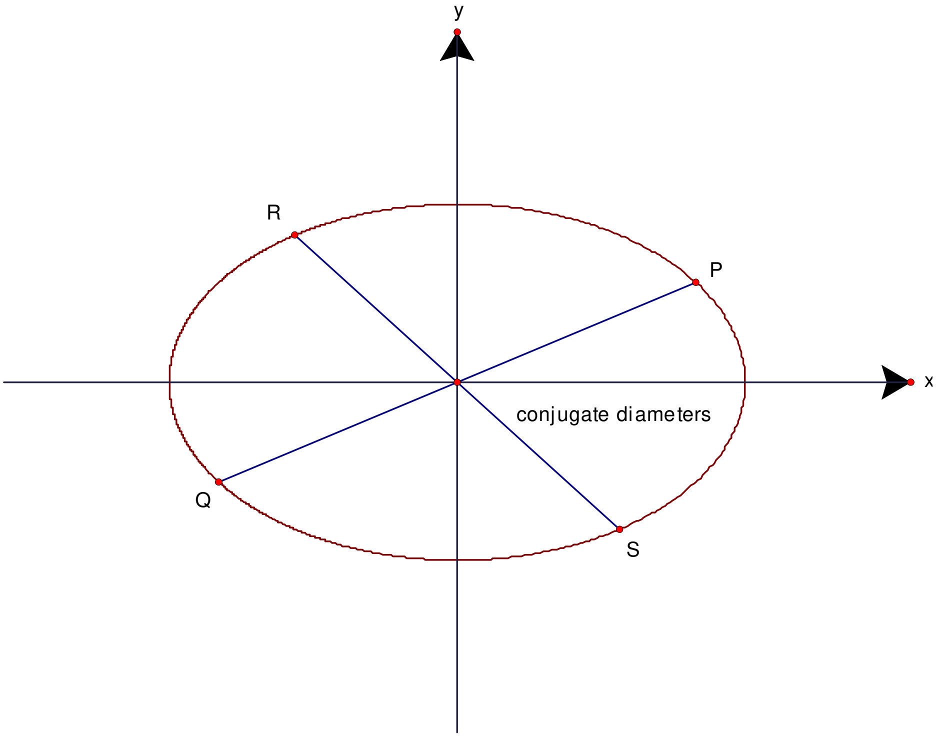
$\dfrac{q}{p}=\dfrac{b^2k}{b^2+a^2m^2}\div\left(-\dfrac{a^2km}{b^2+a^2m^2}\right)=-\dfrac{b^2k}{b^2+a^2m^2}\cdot\dfrac{b^2+a^2m^2}{a^2km}=-\dfrac{b^2}{a^2m}$

$\therefore$ The equation of locus is $\dfrac{y}{x}=-\dfrac{b^2}{a^2m}$

$\Rightarrow y=-\dfrac{b^2}{a^2m}x \cdots\cdots (32)$

This locus is called the **diameter**.

16. Two diameters are in **conjugate** if each diameter bisects chords parallel to the other.
Find the condition for two conjugate diameters.



All diameters has the form $y = mx$.

Let the two diameters be $y = mx$, $y = m'x$ respectively.

From (32), if $y = m'x$ is the diameter which bisects all chords parallel to $y = mx$, then $m' = -\frac{b^2}{a^2 m}$

Hence $mm' = -\frac{b^2}{a^2}$ …… (33)

Suppose PQ, RS are two conjugate diameters. Let the parameters of P and R be θ and φ respectively.

Then ∵ PQ and RS pass through the origin O. (∵ diameters are in the form $y = mx$)

$\therefore PQ$: $y = (\frac{b}{a}\tan\theta)\, x$, RS: $y = (\frac{b}{a}\tan\phi)\, x$

∵ Product of slope $= -\frac{b^2}{a^2}$ (by (33))

$-\frac{b^2}{a^2} = \frac{b^2}{a^2}\tan\theta\tan\phi$

$\tan\theta\tan\phi = -1$

$\phi = \theta + 90°$ …… (34)

17. The sum of square of two conjugate diameters is a constant (= $4a^2 + 4b^2$)

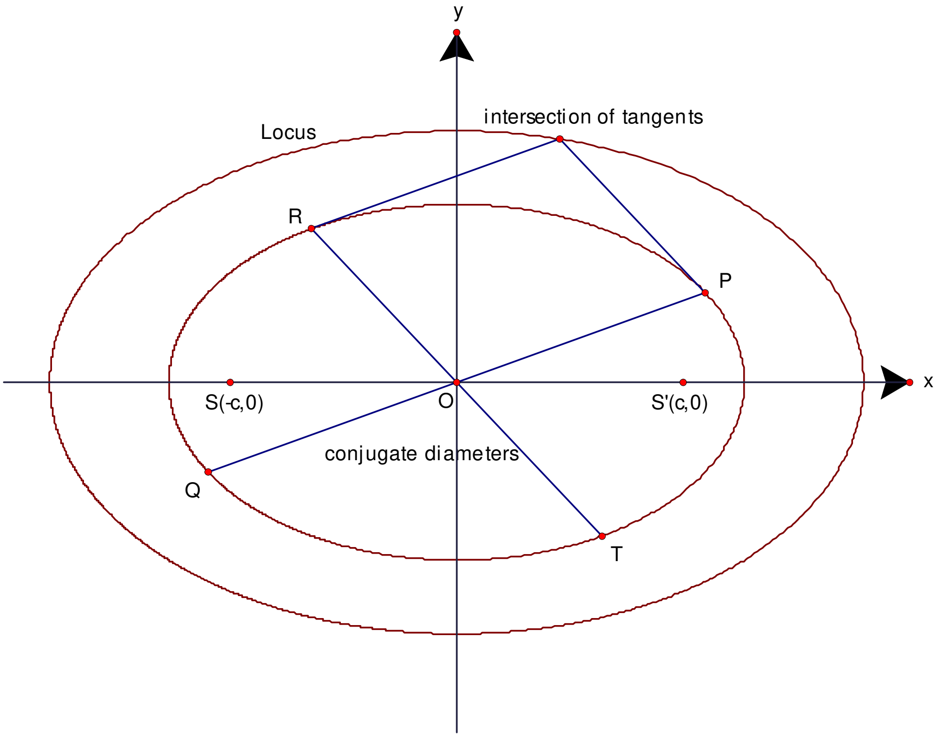
Let PQ and RT be 2 conjugate diameters. The parameters of P and R are θ and θ + 90° respectively.

$PQ^2 + RT^2 = (2OP)^2 + (2OR)^2$

$$= 4[(a\cos\theta)^2 + (b\sin\theta)^2] + 4[(a\cos(\theta + 90°)]^2 + 4[(a\sin(\theta + 90°)]^2$$

$$= 4a^2 + 4b^2 \cdots\cdots (35)$$

18. Find the locus of intersection of tangents of 2 conjugate diameters.



Let PQ, RT be 2 conjugate diameters. The parameters of P and R are θ and θ + 90° respectively.

Equations of tangents: $\begin{cases} \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \\ \frac{x}{a}\cos(\theta + 90^\circ) + \frac{y}{b}\sin(\theta + 90^\circ) = 1 \end{cases} \Rightarrow \begin{cases} \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \\ -\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 1 \end{cases}$

Solving these: $\cos\theta : \sin\theta : 1 = -\frac{x}{a} - \frac{y}{b} : \frac{x}{a} - \frac{y}{b} : -\frac{x^2}{a^2} - \frac{y^2}{b^2}$

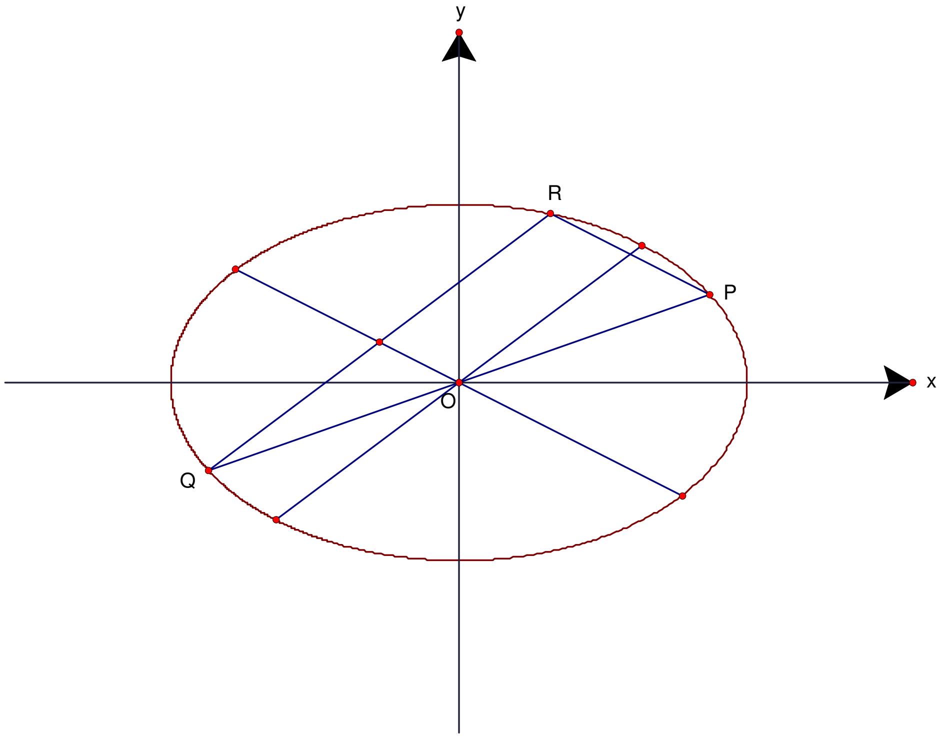
$\because \cos^2\theta + \sin^2\theta = 1$ $\therefore (-\frac{x}{a} - \frac{y}{b})^2 + (\frac{x}{a} - \frac{y}{b})^2 = (-\frac{x^2}{a^2} - \frac{y^2}{b^2})^2$

$2(\frac{x^2}{a^2} + \frac{y^2}{b^2}) = (\frac{x^2}{a^2} + \frac{y^2}{b^2})^2$

$\because \frac{x^2}{a^2} + \frac{y^2}{b^2} \neq 0$, $\therefore$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \cdots\cdots (36)$

The locus is another concentric ellipse.

19. Let *PQ* be any diameter, *R* is a point on the ellipse. Then the diameters parallel to *PR* and *QR* are in conjugate to each other.



Let the parameters of P, Q and R be θ, $\theta + 180°$ and ϕ respectively.

Slope of $PR = \dfrac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}$, slope of $QR = \dfrac{b\sin(180^\circ + \theta) - b\sin\phi}{a\cos(180^\circ + \theta) - a\cos\phi} = \dfrac{-b\sin\theta - b\sin\phi}{-a\cos\theta - a\cos\phi}$

$$\text{Product of slopes} = \frac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}\cdot\frac{b\sin\theta + b\sin\phi}{a\cos\theta + a\cos\phi}$$

$$= \frac{b^2}{a^2}\cdot\frac{(\sin\theta - \sin\phi)(\sin\theta + \sin\phi)}{(\cos\theta - \cos\phi)(\cos\theta + \cos\phi)}$$

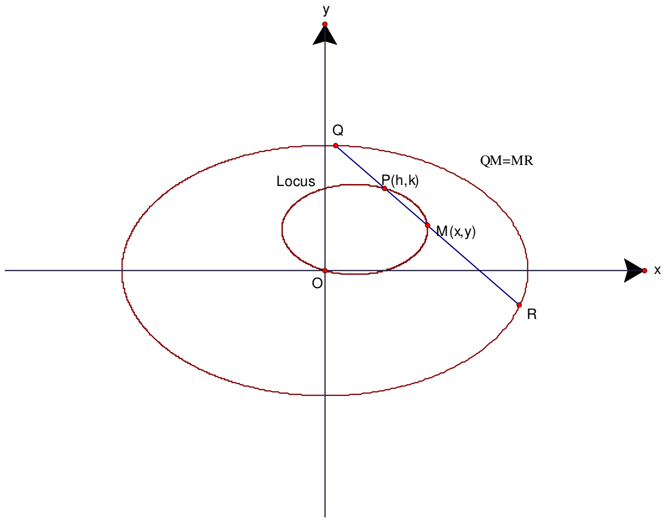
$$= \frac{b^2}{a^2}\cdot\frac{\left(2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right)\left(2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right)}{\left(-2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right)\left(2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right)}$$

$$= -\frac{b^2}{a^2}$$

$\therefore$ By (33), the diameters parallel to PR and QR are conjugate to each other.

20. *P* is a fixed point. Find the locus of mid-point of chords through *P*.

Let the mid point of chord through $P = (x, y)$.



Slope of $OM \times$ slope of $MP = -\frac{b^2}{a^2}$

$$\frac{y}{x}\cdot\frac{y-k}{x-h} = -\frac{b^2}{a^2}$$

$$a^2y^2 - a^2ky + b^2x^2 - b^2hx = 0$$

$$a^2(y - \frac{k}{2})^2 + b^2(x - \frac{h}{2})^2 = \frac{a^2k^2 + b^2h^2}{4} \cdots(37)$$

It is an ellipse whose centre is the mid-point of OP.

Method 2 Let the equation of the straight line be $y - k = m(x - h) \Rightarrow y = mx + k - mh\cdots(38)$

It intersects the ellipse at $Q(x_1, y_1)$, $R(x_2, y_2)$.

Sub. (38) into E: $b^2x^2 + a^2(mx + k - mh)^2 = a^2b^2$

$(a^2m^2 + b^2)x^2 + 2a^2m(k - mh)x + a^2[(k - mh)^2 - b^2] = 0$, roots x_1, x_2

$$\frac{x_1 + x_2}{2} = -\frac{2a^2m(k-mh)}{2(a^2m^2+b^2)} = \frac{a^2m(mh-k)}{a^2m^2+b^2}$$

$$\therefore \quad \frac{y_1 + y_2}{2} = \frac{1}{2}[m(x_1 + x_2)] + k - mh = m\cdot\frac{a^2m(mh-k)}{a^2m^2+b^2} + \frac{(k-mh)(a^2m^2+b^2)}{(a^2m^2+b^2)} = -\frac{b^2(mh-k)}{a^2m^2+b^2}$$

Let the mid-point be $M(x, y)$.

$$x = \frac{a^2m(mh-k)}{a^2m^2+b^2} \quad \cdots\cdots (39)$$

$$y = -\frac{b^2(mh-k)}{a^2m^2+b^2} \quad \cdots\cdots (40)$$

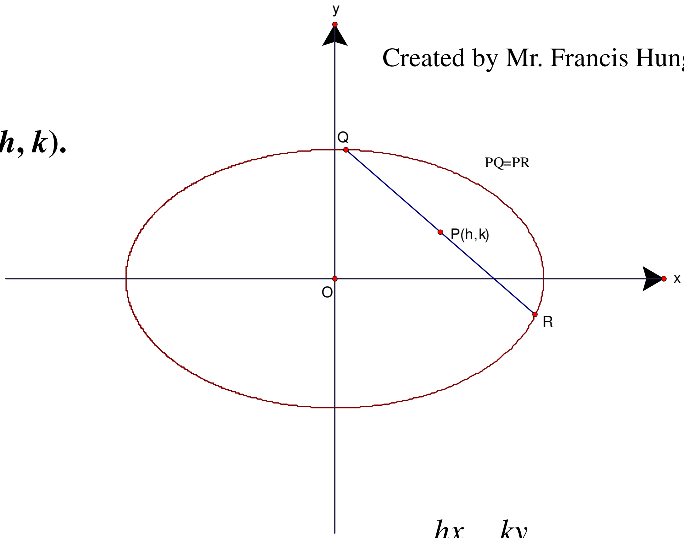
(39)÷(40) $\frac{x}{y} = -\frac{a^2m}{b^2} \Rightarrow m = -\frac{b^2x}{a^2y} \quad \cdots\cdots (41)$

Sub. into (40): $y = -\frac{b^2\left(-\frac{b^2x}{a^2y}\cdot h - k\right)}{a^2\cdot\frac{b^4x^2}{a^4y^2} + b^2} = -\frac{b^2(-b^2hxy - a^2ky^2)}{b^4x^2 + a^2b^2y^2} = \frac{y(a^2ky + b^2hx)}{b^2x^2 + a^2y^2}$

$$b^2x^2 + a^2y^2 = a^2ky + b^2hx$$

$$b^2\left(x - \frac{h}{2}\right)^2 + a^2\left(y - \frac{k}{2}\right)^2 = b^2\left(\frac{h}{2}\right)^2 + a^2\left(\frac{k}{2}\right)^2$$

21. Find the chord with given mid-point (h, k).



Slope of $OP = \frac{k}{h}$ $\therefore$ slope of chord $= -\frac{b^2}{a^2} \times \frac{h}{k}$

Equation of chord: $\frac{y-k}{x-h} = -\frac{b^2 h}{a^2 k}$

$a^2ky - a^2k^2 + b^2hx - b^2h^2 = 0$

$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ ...... (42)

If (h, k) lies on the ellipse ($\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$), then the chord will become a tangent $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$

Method 2

Let the equation of the straight line be $y - k = m(x - h) \Rightarrow y = mx + k - mh$ ...... (38)

It intersects the ellipse at $Q(x_1, y_1)$, $R(x_2, y_2)$.
Sub. (38) into E: $b^2x^2 + a^2(mx + k - mh)^2 = a^2b^2$
$(a^2m^2 + b^2)x^2 + 2a^2m(k - mh)x + a^2[(k - mh)^2 - b^2] = 0$, roots x_1, x_2

$\frac{x_1 + x_2}{2} = -\frac{2a^2m(k-mh)}{2(a^2m^2+b^2)} = \frac{a^2m(mh-k)}{a^2m^2+b^2} = h$ ...... (39)

$\frac{y_1 + y_2}{2} = \frac{1}{2}[m(x_1+x_2)] + k - mh = m \cdot \frac{a^2m(mh-k)}{a^2m^2+b^2} + \frac{(k-mh)(a^2m^2+b^2)}{(a^2m^2+b^2)} = -\frac{b^2(mh-k)}{a^2m^2+b^2} = k$ ......(40)

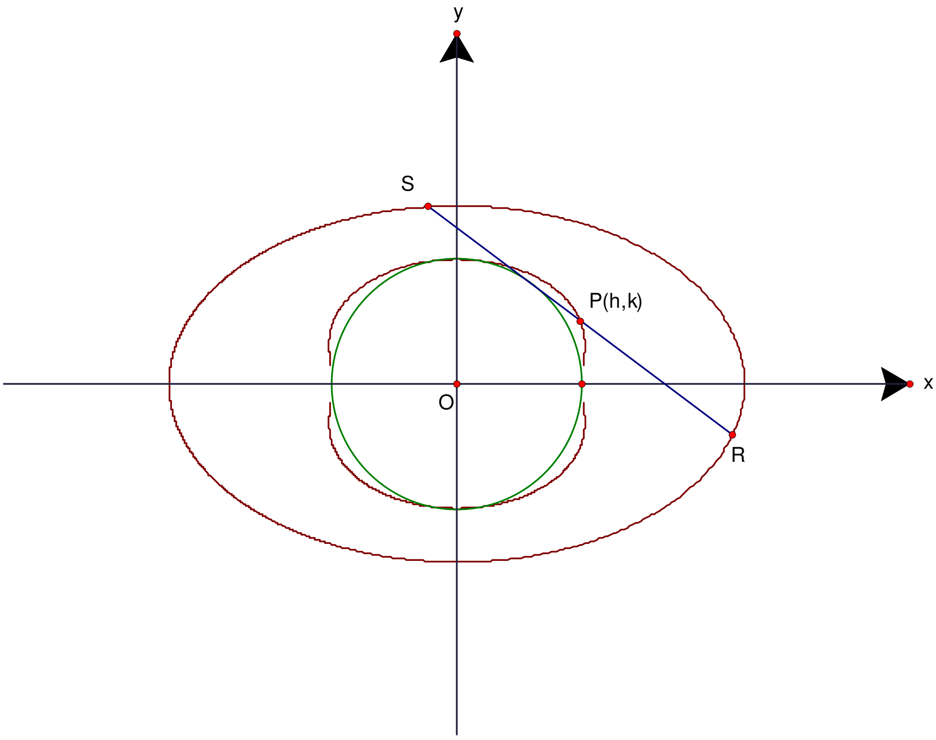
(39)÷(40) $\frac{h}{k} = -\frac{a^2m}{b^2} \Rightarrow m = -\frac{b^2h}{a^2k}$ ...... (41)

Sub. into (38): $y - k = -\frac{b^2h}{a^2k}(x - h)$

$\frac{k}{b^2}(y-k) = -\frac{h}{a^2}(x-h)$

$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ ...... (42)

22. Find the locus of mid-point of chords which touches the circle $x^2 + y^2 = r^2$, where $r < b < a$



Let $P(h, k)$ be the mid-point.

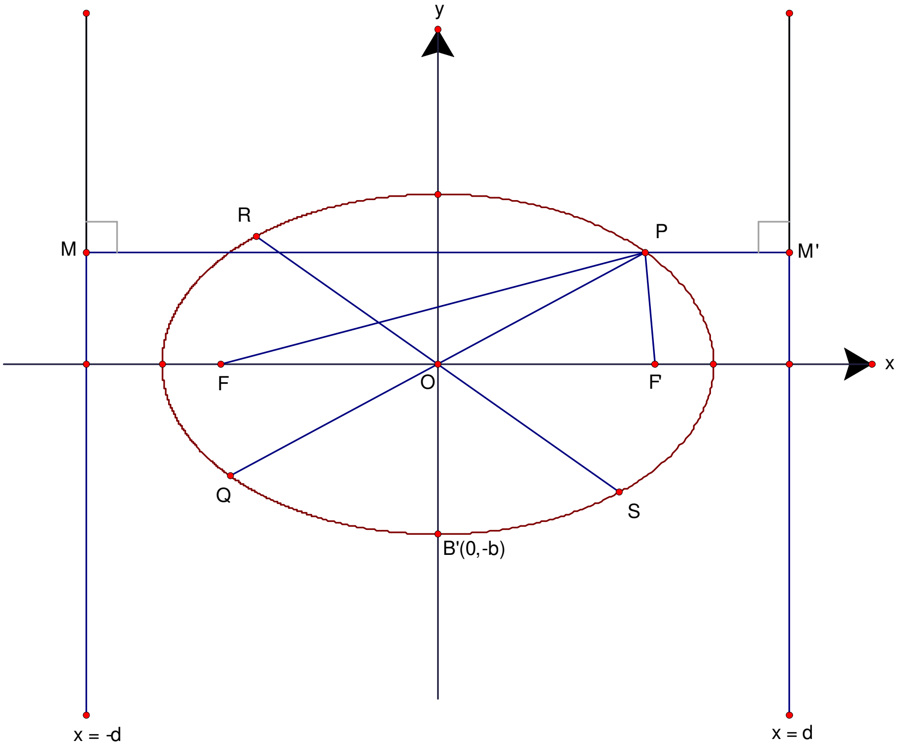
Equation of chord: $\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$ …… (42)

$\because$ It touches the circle, $\therefore$ distance from origin to the chord = r

$$\left|\frac{\frac{h^2}{a^2}+\frac{k^2}{b^2}}{\sqrt{\left(\frac{h}{a^2}\right)^2+\left(\frac{k}{b^2}\right)^2}}\right|=r$$

$\therefore$ Locus of mid-point is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=r^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$ …… (43)

23. Suppose PQ and RS are two conjugate diameters, foci = F, F'. Show that $FP \cdot F'P = OR^2$.



Let the parameters of P and R be θ and θ + 90°.

Let M, M' be the feet of perpendiculars from P onto both directrices.

$PM = a\cos\theta + \dfrac{a}{e}$, $PM' = \dfrac{a}{e} - a\cos\theta$

$FP = e\,PM = a(e\cos\theta + 1)$, $F'P = e\,PM' = a(1 - e\cos\theta)$

$FP \cdot F'P = a^2(1 - e^2\cos^2\theta)$

$= a^2 - (a^2 - b^2)\cos^2\theta$

$= (a\sin\theta)^2 + (b\cos\theta)^2$ ⋯⋯ (44)

$OR^2 = (a\sin\theta)^2 + (b\cos\theta)^2$

$\therefore FP \cdot F'P = OR^2$ ⋯⋯ (45)

24. Find the equations of conjugate diameters of equal length.

If the diameters are equal in length, then

$a^2 \cos^2 \theta + b^2 \sin^2 \theta = a^2 \sin^2 \theta + b^2 \cos^2 \theta$

$a^2 \cos 2\theta = b^2 \cos 2\theta$

$(a^2 - b^2) \cos 2\theta = 0$

$\because a \neq b$, $\therefore \cos 2\theta = 0$

$2\theta = 90°, 270°, 450°, 630°$

$\theta = 45°, 135°, 225°, 315°$

$\therefore$ Equations of diameters: $\begin{cases} y = \frac{b}{a} \tan 45^\circ x \\ y = \frac{b}{a} \tan 135^\circ x \end{cases}$

$\Rightarrow \begin{cases} \frac{x}{a} - \frac{y}{b} = 0 \\ \frac{x}{a} + \frac{y}{b} = 0 \end{cases}$ $\cdots\cdots$ (46)

25. Equation of normal

Suppose the equation of tangent at (x_0, y_0) on the ellipse is (22): $\frac{x_0 x}{a^2}+\frac{y_0 y}{b^2}=1$

The slope of tangent $=-\frac{x_0}{a^2}\cdot\frac{b^2}{y_0}=-\frac{b^2 x_0}{a^2 y_0}$

$\therefore$ Slope of normal $=\frac{a^2 y_0}{b^2 x_0}$

Equation of normal

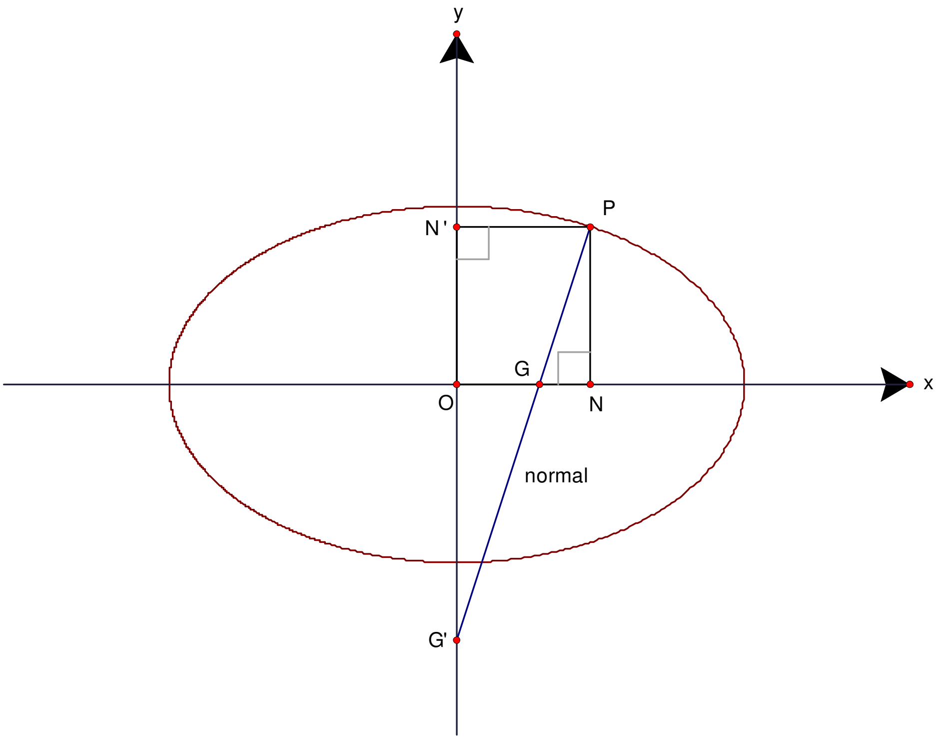
$$\frac{y-y_0}{x-x_0}=\frac{a^2 y_0}{b^2 x_0}$$

$$(b^2x_0)y-b^2x_0y_0=(a^2y_0)x-a^2x_0y_0$$

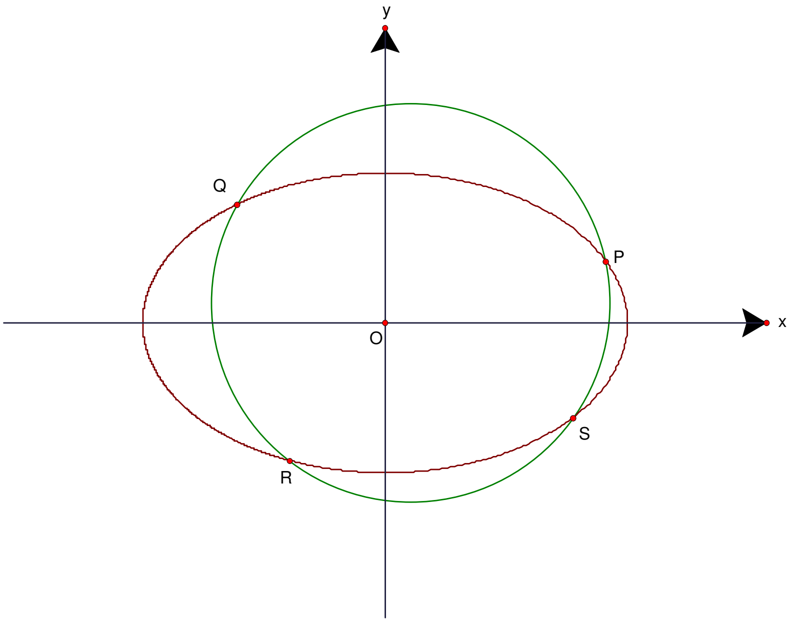
$$\frac{a^2x}{x_0}-\frac{b^2y}{y_0}=a^2-b^2 \quad \cdots\cdots (47)$$

In parametric form: $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \quad \cdots\cdots (48)$

Exercise 1 The normal at P meets the major and minor axes at G and G' respectively. N and N' are the feet of perpendiculars from P to the major and minor axes. Prove that $OG : ON = e^2 : 1$, and that $OG' : ON' = c^2 : b^2$



26. Concyclic points.



Suppose a circle intersects an ellipse at P, Q, R and S. Let the parameters be α, β, θ and ϕ respectively.

To find the condition for which α, β, θ and ϕ to be concyclic.

The equation of chords joining α, β, and θ, ϕ are given by (20):

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\beta-\alpha}{2}\text{; and}$$

$$\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}\quad\text{respectively.}$$

Let $U=\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\cos\frac{\beta-\alpha}{2}$;

$V=\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}-\cos\frac{\theta-\phi}{2}$

Consider the equation: $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-1+kUV=0$, where k is a constant.

This is a second degree equation which contains α, β, θ and ϕ.

It is a circle if and only if coefficient of xy = 0 and coefficient of x^2 = coefficient of y^2

$$\therefore \frac{1}{ab}\left[\sin\frac{\theta+\phi}{2}\cos\frac{\alpha+\beta}{2}+\cos\frac{\theta+\phi}{2}\sin\frac{\alpha+\beta}{2}\right]=0 \cdots\cdots (49)$$

$$\text{and}\quad \frac{1}{a^2}\left(1+k\cos\frac{\theta+\phi}{2}\cos\frac{\alpha+\beta}{2}\right)=\frac{1}{b^2}\left(1+k\sin\frac{\theta+\phi}{2}\sin\frac{\alpha+\beta}{2}\right) \cdots\cdots (50)$$

$(49) \Rightarrow \sin\frac{\alpha+\beta+\theta+\phi}{2}=0$

$\Rightarrow \alpha+\beta+\theta+\phi=360°m$, where m is an integer. (Multiples of 360°)

This is the required condition.

(50) determines the value of k.

Method 2 Put (a cos θ, b sin θ) into the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

$a^2 \cos^2 \theta + b^2 \sin^2 \theta + 2ag \cos \theta + 2bf \sin \theta + c = 0$

Use the circular function of t: $\cos \theta = \dfrac{1-t^2}{1+t^2}$, $\sin \theta = \dfrac{2t}{1+t^2}$

$$a^2 \cdot \left(\frac{1-t^2}{1+t^2}\right)^2 + b^2\left(\frac{2t}{1+t^2}\right)^2 + 2ag \cdot \frac{1-t^2}{1+t^2} + 2bf \cdot \frac{2t}{1+t^2} + c = 0$$

$a^2(1 - 2t^2 + t^4) + 4b^2t^2 + 2ag(1 - t^4) + 4bft(1 + t^2) + c(1 + 2t^2 + t^4) = 0$

$(a^2 - 2ag + c)t^4 + 4bft^3 + (-2a^2 + 4b^2 + 2c)t^2 + 4bft + (a^2 + 2ag + c) = 0$

This is a polynomial equation in t of degree 4, which have 4 roots t_1, t_2, t_3, t_4.

$t_1 = \tan\dfrac{\alpha}{2}$, $t_2 = \tan\dfrac{\beta}{2}$, $t_3 = \dfrac{\theta}{2}$, $t_4 = \dfrac{\phi}{2}$

Using the relation between the roots and coefficients:

$$\sum t_i = -\frac{4bf}{a^2 - 2ag + c},\quad \sum_{i<j} t_i t_j = \frac{-2a^2 + 4b^2 + 2c}{a^2 - 2ag + c},\quad \sum_{i<j<k} t_i t_j t_k = -\frac{4bf}{a^2 - 2ag + c},\quad t_1 t_2 t_3 t_4 = \frac{a^2 + 2ag + c}{a^2 - 2ag + c}$$

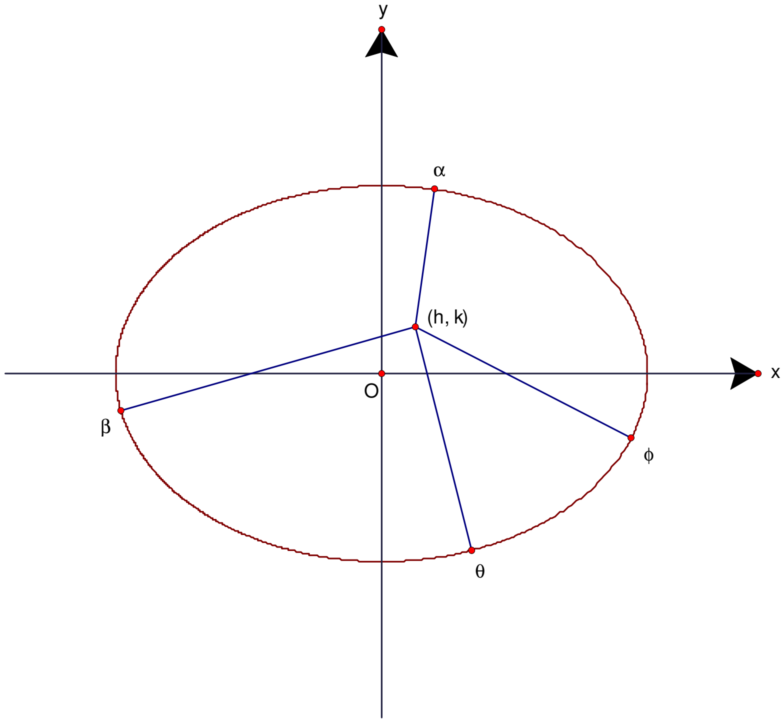
$$\tan\frac{\alpha+\beta+\theta+\phi}{2} = \frac{\sum t_i - \sum t_i t_j t_k}{1 - \sum t_i t_j + t_1 t_2 t_3 t_4} = \frac{-\frac{4bf}{a^2 - 2ag + c} - \left(-\frac{4bf}{a^2 - 2ag + c}\right)}{1 - \sum t_i t_j + t_1 t_2 t_3 t_4} = 0$$

$$\frac{\alpha+\beta+\theta+\phi}{2} = 180°m$$

$\alpha + \beta + \theta + \phi = 360°m$, where m is an integer. (Multiples of 360°)

27. Condition for Conormal points

Normals can be drawn through a point (h, k) and that if the parameters of the feet of normals are α, β, θ and ϕ, then $\alpha + \beta + \theta + \phi = 180°(2m + 1)$, where m is an integer.



Equation of normal is (48): $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$

$\because$ they meet at (h, k), $\therefore$ $\dfrac{ah}{\cos\theta} - \dfrac{bk}{\sin\theta} = a^2 - b^2$

Using the formula $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$

Sub. into (48): $\dfrac{ah(1+t^2)}{1-t^2} - \dfrac{bk(1+t^2)}{2t} = a^2 - b^2$

$ah(1 + t^2)\cdot 2t - bk(1 - t^4) = (a^2 - b^2)\cdot 2t\cdot(1 - t^2)$

$bkt^4 + 2aht^3 + 2aht - bk = 2(a^2 - b^2)t - 2(a^2 - b^2)t^3$

$bkt^4 + 2(ah + a^2 - b^2)t^3 + 2(ah - a^2 + b^2)t - bk = 0 \cdots\cdots (51)$

Since it is an equation in t of degree 4, $\therefore$there are 4 roots t_1, t_2, t_3, t_4 (real or complex).

$t_1 = \tan\dfrac{\alpha}{2}$, $t_2 = \tan\dfrac{\beta}{2}$, $t_3 = \tan\dfrac{\theta}{2}$, $t_4 = \tan\dfrac{\phi}{2}$.

$t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \dfrac{\text{coefficent of } t^2}{\text{coefficient of } t^4} = 0$, $t_1t_2t_3t_4 = \dfrac{\text{coefficent of } t^0}{\text{coefficient of } t^4} = -1$

$$\tan\frac{\alpha+\beta+\theta+\phi}{2} = \frac{\sum t_i - \sum_{i<j<<k} t_i t_j t_k}{1 - \sum_{i<j} t_i t_j + t_1 t_2 t_3 t_4}$$

Since the denominator vanish, $\therefore$ $\alpha + \beta + \theta + \phi = 180°(2m + 1)$, where m is an integer.

28. Suppose α, β, θ are the parameters of the feet of 3 concurrent normals of an ellipse $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$, prove that sin(α + β) + sin(β + θ) + sin(α + θ) = 0

Equation of normal is: $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ ⋯⋯ (48)

$(a\sin\theta)x-(b\cos\theta)y=(a^2-b^2)\sin\theta\cos\theta$

Suppose they meet at (h, k), then

$(a\sin\theta)h-(b\cos\theta)k=(a^2-b^2)\sin\theta\cos\theta$ ⋯⋯ (52)

$a^2h^2\sin^2\theta=\cos^2\theta[bk+(a^2-b^2)\sin\theta]^2$

$a^2h^2\sin^2\theta=(1-\sin^2\theta)[b^2k^2+(a^2-b^2)^2\sin^2\theta+2bk(a^2-b^2)\sin\theta]$

$(a^2-b^2)^2\sin^4\theta+2bk(a^2-b^2)\sin^3\theta+\ldots=0$

This is an equation in sin θ with degree = 4, there are four roots to the equation.

Let the four roots be sin α, sin β, sin θ and sin ϕ.

$\therefore\ \sin\alpha+\sin\beta+\sin\theta+\sin\phi=-\frac{\text{coefficient of }\sin^3\theta}{\text{coefficient of }\sin^4\theta}=-\frac{2bk}{a^2-b^2}$ ⋯⋯ (53)

On the other hand, by (52): $bk\cos\theta=[ah-(a^2-b^2)\cos\theta]\sin\theta$

$b^2k^2\cos^2\theta=(1-\cos^2\theta)[a^2h^2-2ah(a^2-b^2)\cos\theta+(a^2-b^2)^2\cos^2\theta]$

$(a^2-b^2)^2\cos^4\theta-2ah(a^2-b^2)\cos^3\theta+\ldots=0$

This is an equation in cos θ with degree = 4, the four roots are cos α, cos β, cos θ and cos ϕ.

$\therefore\ \cos\alpha+\cos\beta+\cos\theta+\cos\phi=\frac{2ah}{a^2-b^2}$ ⋯⋯ (54)

(53): $\sin\alpha+\sin\beta+\sin\theta=-\frac{2bk}{a^2-b^2}-\sin\phi$ ⋯⋯ (55)

(54): $\cos\alpha+\cos\beta+\cos\theta=\frac{2ah}{a^2-b^2}-\cos\phi$ ⋯⋯ (56)

Multiply together: $\sin(\alpha+\beta)+\sin(\beta+\theta)+\sin(\alpha+\theta)+\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\theta\cos\theta$

$$=-\frac{4abhk}{(a^2-b^2)^2}+\frac{2bk}{a^2-b^2}\cos\phi-\frac{2ah}{a^2-b^2}\sin\phi+\sin\phi\cos\phi \quad\cdots\cdots (57)$$

Recall in (52), $\sin\theta\cos\theta=\frac{ah\sin\theta-bk\cos\theta}{a^2-b^2}$, θ can be replaced by α, β, θ and ϕ.

Sub. these into (57):

$$\sin(\alpha+\beta)+\sin(\beta+\theta)+\sin(\alpha+\theta)+\frac{ah\sin\alpha-bk\cos\alpha}{a^2-b^2}+\frac{ah\sin\beta-bk\cos\beta}{a^2-b^2}+\frac{ah\sin\theta-bk\cos\theta}{a^2-b^2}$$

$$=-\frac{4abhk}{(a^2-b^2)^2}+\frac{2bk}{a^2-b^2}\cos\phi-\frac{2ah}{a^2-b^2}\sin\phi+\frac{ah\sin\phi-bk\cos\phi}{a^2-b^2}$$

$\sin(\alpha+\beta)+\sin(\beta+\theta)+\sin(\alpha+\theta)+\frac{ah(\sin\alpha+\sin\beta+\sin\theta)-bk(\cos\alpha+\cos\beta+\cos\phi)}{a^2-b^2}$

$=-\frac{4abhk}{(a^2-b^2)^2}+\frac{2bk}{a^2-b^2}\cos\phi-\frac{2ah}{a^2-b^2}\sin\phi+\frac{ah\sin\phi-bk\cos\phi}{a^2-b^2}$

By (55) and (56), $\sin(\alpha+\beta)+\sin(\beta+\theta)+\sin(\alpha+\theta)+\frac{ah\left(-\frac{2bk}{a^2-b^2}-\sin\phi\right)-bk\left(\frac{2ah}{a^2-b^2}-\cos\phi\right)}{a^2-b^2}$

$=-\frac{4abhk}{(a^2-b^2)^2}+\frac{2bk}{a^2-b^2}\cos\phi-\frac{2ah}{a^2-b^2}\sin\phi+\frac{ah\sin\phi-bk\cos\phi}{a^2-b^2}$

$\therefore \sin(\alpha+\beta)+\sin(\beta+\theta)+\sin(\alpha+\theta)\ -\frac{2abhk}{(a^2-b^2)^2}-\frac{2abhk}{(a^2-b^2)^2}-\frac{ah\sin\phi}{a^2-b^2}+\frac{bk\cos\phi}{a^2-b^2}$

$=-\frac{4abhk}{(a^2-b^2)^2}+\frac{bk}{a^2-b^2}\cos\phi-\frac{ah}{a^2-b^2}\sin\phi$

$\therefore \sin(\alpha+\beta)+\sin(\beta+\theta)+\sin(\alpha+\theta)=0$