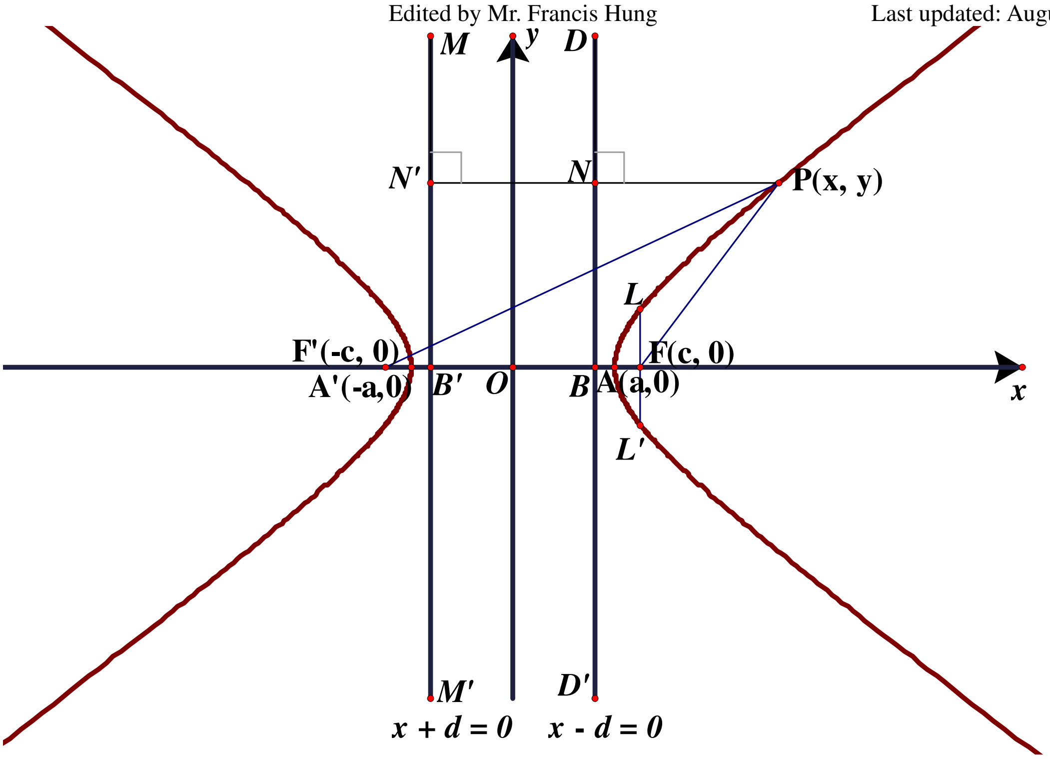
# Hyperbola

Reference: Advanced Level Pure Mathematics by S.L. Green (4th Edition) Chapter VII p.91 - p.102

Edited by Mr. Francis Hung Last updated: August 30, 2021




1. **Definition** $F$ is the **focus**, $x$-axis is **the principal axis**. $DD'$ is **the directrix**.

A variable point $P(x, y)$ moves so that $\frac{PF}{PN} = e$ (a constant called the '**eccentricity**' > 1).

where $N$ is the foot of perpendicular from $P$ onto $DD'$.

In particular, when $P$ moves to $A$ (between $B$ and $F$) on the principal axis, $AF = eAB$

Produce $FB$ further to a point $A'$ such that $A'F = eA'B$; then $A'$ is on the curve.

(Note that $A$ and $A'$ are on the opposite sides of $DD'$.)

Bisect $AA'$ at $O(0, 0)$ (called the **centre**) and let $AA' = 2a$, then $A = (a, 0)$, $A' = (-a, 0)$.

Let $DD'$ be $x = d$.

$c - a = e(a - d)$ ⋯⋯ (1)

$c + a = e(a + d)$ ⋯⋯ (2)

[(1) + (2)]÷2: $c = ae$ ⋯⋯ (3)

[(2) − (1)]÷2: $a = de$ ⋯⋯ (4)

Now let $P = (x, y)$, $PF = ePN$

$\sqrt{(x-c)^2 + y^2} = e(x-d)$

$(x - ae)^2 + y^2 = e^2(x - \frac{a}{e})^2$ by (3) and (4)

$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$

$(e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$

$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$ (Given that $e > 1$ ∴ $e^2 - 1 > 0$)

Let $b = a\sqrt{e^2-1} > 0$, then the equation of the locus becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ⋯⋯ (5)

and $b^2 = (ae)^2 - a^2 \Rightarrow b^2 = c^2 - a^2$

$\Rightarrow a^2 + b^2 = c^2$ ⋯⋯ (6)

Replace $x$ by $-x$ and $y$ by $-y$ in (5) respectively, there is no change.

∴ **The locus is symmetrical about the $x$-axis and $y$-axis** ⋯⋯ (7)

2. For any point $P(x, y)$ on the hyperbola, $P'(-x, y)$ is the image of $P$ reflected along $y$-axis.

   By (7), the locus $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is symmetric about $y$-axis

   $\therefore$ $P'(-x, y)$ also lies on the curve $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

   Let $F'(-c, 0)$ be the image of $F(c, 0)$ reflected along $y$-axis.

   $MM'$: $x = -d$ is the image of the directrix $DD'$ reflected along $y$-axis.

   Then $\frac{F'P'}{P'N'}=e$ (all dashes are images reflected along $y$-axis)

   $P$ is the image of $P'$, which lies on the curve.

   $\therefore$ $\frac{F'P}{PN'}=e$ for any point $P$ on the ellipse.

   $\therefore$ There are two **foci** $F(a, 0)$, $F'(-a, 0)$ and two **directrices** $DD'$: $x - d = 0$ and $EE'$: $x + d = 0$.

3. The equation gives $\frac{x^2}{a^2}=1+\frac{y^2}{b^2}\ge 1$

   From which $x^2 \ge a^2 \Rightarrow x \le -a$ or $x \ge a$. $A(a, 0)$ and $A'(-a, 0)$ are the vertices of the hyperbola.

   It follows that there is no point between $x = -a$ to $x = a$.

   The curve has two distinct **branches**.

4. The **latus rectum** is the line segment $LL'$ through $F(c, 0)$ and is perpendicular to the principal axis. To find the coordinates of $L$ and $L'$: $\begin{cases}\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\\ x=ae\end{cases}$

   By substitution, $\frac{a^2e^2}{a^2}-\frac{y^2}{b^2}=1$

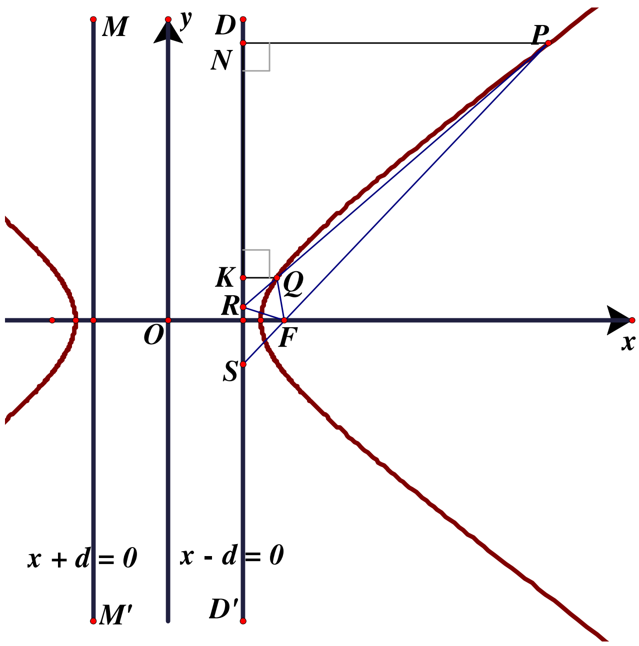
   $y=\pm b\sqrt{e^2-1}$

   $\therefore$ $L=\left(ae, b\sqrt{e^2-1}\right)$, $L'=\left(ae, -b\sqrt{e^2-1}\right)$

   $LL'=2b\sqrt{e^2-1}=2b\cdot\frac{b}{a}$

   $\Rightarrow LL'=\frac{2b^2}{a}$ …… (8)

## 5. Geometrical property




Let $P$, $Q$ be two points **on the same branch** of a hyperbola.

$N$ is the foot of perpendicular drawn from $P$ onto the line $DD'$. $K$ is the foot of perpendicular drawn from $Q$ onto the line $DD'$. The chord $PQ$ is produced to $R$ on $DD'$. Join $RF$, the line joining $P$, $F$ is produced to meet $DD'$ at $S$.

Let $\angle QFR = \alpha$, $\angle RFS = \beta$, $\angle QRF = \theta$

then $PF = ePN$, $QF = eQK$ by definition

$\Rightarrow \dfrac{PF}{QF} = \dfrac{PN}{QK}$ …… (9)

$\Delta PNR \sim \Delta QKR$ (equiangular)

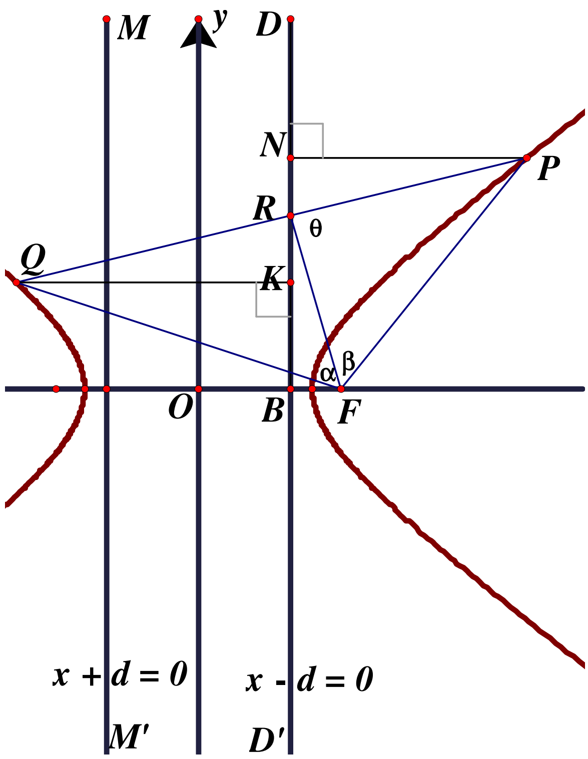
$\Rightarrow \dfrac{PF}{QF} = \dfrac{PN}{QK} = \dfrac{PR}{QR} \Rightarrow \dfrac{PF}{QF} = \dfrac{PR}{QR}$ …… (10)

Apply sine law on $\Delta PRF$: $\dfrac{PF}{PR} = \dfrac{\sin\theta}{\sin\beta}$ …… (11)

$\Delta FQR$, $\dfrac{QF}{QR} = \dfrac{\sin\theta}{\sin\alpha}$ …… (12)

(11)÷(12) ÷(10) $\Rightarrow \alpha = \beta$

$\therefore$ ***RF*** is the **exterior angle bisector** of $\angle PFQ$.




Let $P$, $Q$ be two points **on the different branches** of a hyperbola.

$N$ is the foot of perpendicular drawn from $P$ onto the line $DD'$. $K$ is the foot of perpendicular drawn from $Q$ onto the line $DD'$.

The chord $PQ$ cuts $DD'$ at $R$. Join $RF$.

Let $\angle QFR = \alpha$, $\angle PFR = \beta$, $\angle PRF = \theta$

then $PF = ePN$, $QF = eQK$ by definition

$\Rightarrow \dfrac{PF}{QF} = \dfrac{PN}{QK}$ …… (13)

$\Delta PNR \sim \Delta QKR$ (equiangular)

$\Rightarrow \dfrac{PF}{QF} = \dfrac{PN}{QK} = \dfrac{PR}{QR} \Rightarrow \dfrac{PF}{QF} = \dfrac{PR}{QR}$ …… (14)

Sine law on $\Delta PRF$: $\dfrac{PF}{PR} = \dfrac{\sin\theta}{\sin\beta}$ …… (15)

$\Delta FQR$, $\dfrac{QF}{QR} = \dfrac{\sin\theta}{\sin\alpha}$ …… (16)

(15)÷(16) ÷(14) $\Rightarrow \alpha = \beta$

$\therefore$ ***RF*** is the **interior angle bisector** of $\angle PFQ$.

6. **<u>Variation</u>** $O(0, 0)$ is the centre of the curve.

The curve $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is a hyperbola which opens upward and downward.

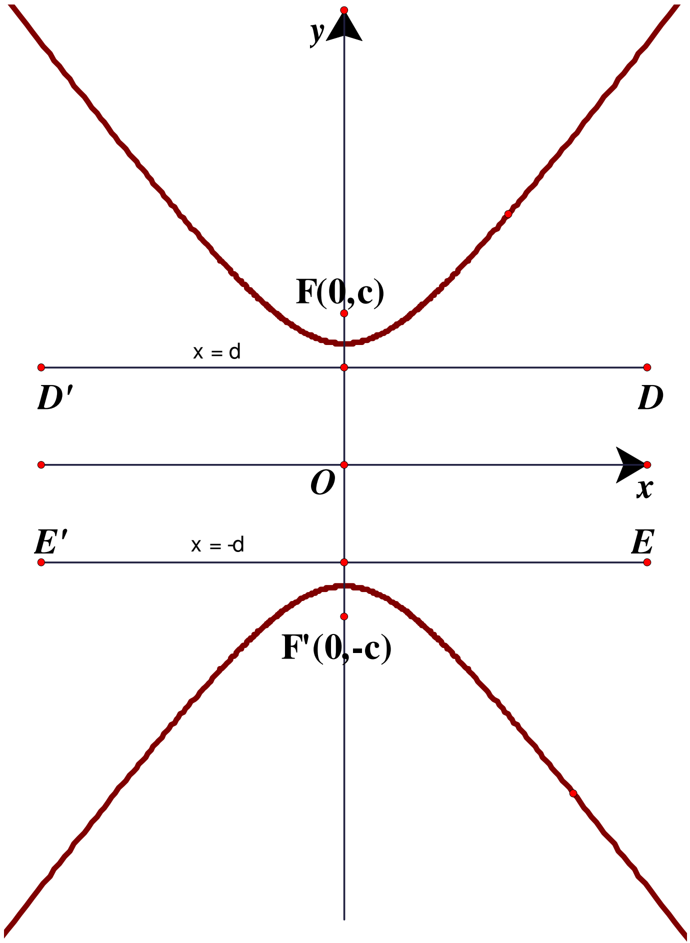
The foci are at $F(0, c)$, $F'(0, -c)$.

$c = ae$ …… (3)

The directrix is $DD'$ whose equation is $y = d$ and $EE'$ whose equation is $y = -d$,

where $d=\frac{a}{e}$ ⋯⋯ (4)

It is clear that $a^2 + b^2 = c^2$ ⋯⋯ (6)




If **the centre of a hyperbola is translated to $V(h, k)$,** then its new equation is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \quad \cdots\cdots (17)$$

**Example 1** Consider the equation $2x^2 + 4x - 3y^2 + 12y - 4 = 0$

It can be transformed into standard equation <u>by completing the square</u>:

$2(x^2 + 2x) - 3(y^2 - 4y) - 4 = 0$

$2(x^2 + 2x + 1) - 3(y^2 - 4y + 4) + 6 = 0$

$3(y - 2)^2 - 2(x + 1)^2 = 6$

$$\frac{(y-2)^2}{\sqrt{2}^2}-\frac{(x+1)^2}{\sqrt{3}^2}=1$$

It is a hyperbola which opens upward and downward with centre at $(-1, 2)$, $a=\sqrt{2}$, $b=\sqrt{3}$.

$c^2 = a^2 + b^2 = 2 + 3 = 5 \Rightarrow c=\sqrt{5}$

$e=\frac{c}{a}=\sqrt{\frac{5}{2}}$, $d=\frac{a}{e}=\frac{2}{\sqrt{5}}$.

7. **Parametric equations**

$\begin{cases} x=a\sec\theta \\ y=b\tan\theta \end{cases}$ ⋯⋯ (18) are the parametric equations of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where θ is a parameter.

Similarly $\begin{cases} x=b\tan\theta \\ y=a\sec\theta \end{cases}$ ⋯⋯ (19) are the parametric equations of $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

8. **Equation of chords using parameters**.

Let $A(a\sec\alpha, b\tan\alpha)$, $B(a\sec\beta, b\tan\beta)$ be 2 points of the hyperbola. Then the equation of $AB$ is $\dfrac{y-b\tan\beta}{x-a\sec\beta}=\dfrac{b\tan\alpha-b\tan\beta}{a\sec\alpha-a\sec\beta}$

$$\frac{y-b\tan\beta}{x-a\sec\beta}=\frac{b\left(\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}\right)\cdot\cos\alpha\cos\beta}{a\left(\frac{1}{\cos\alpha}-\frac{1}{\cos\beta}\right)\cdot\cos\alpha\cos\beta}$$

$$\frac{y-b\tan\beta}{x-a\sec\beta}=\frac{b(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{a(\cos\beta-\cos\alpha)}$$

$$\frac{y-b\tan\beta}{x-a\sec\beta}=\frac{b\sin(\alpha-\beta)}{2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\frac{y-b\tan\beta}{x-a\sec\beta}=\frac{2b\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}}{2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\frac{y-b\tan\beta}{x-a\sec\beta}=\frac{b\cos\frac{\alpha-\beta}{2}}{a\sin\frac{\alpha+\beta}{2}}$$

$$\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\frac{b\tan\beta}{b}\sin\frac{\alpha+\beta}{2}=\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{a\sec\beta}{a}\cos\frac{\alpha-\beta}{2}$$

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\sec\beta\cos\frac{\alpha-\beta}{2}-\tan\beta\sin\frac{\alpha+\beta}{2}$$

$$=\sec\beta\left(\cos\frac{\alpha-\beta}{2}-\sin\beta\sin\frac{\alpha+\beta}{2}\right)$$

$$=\sec\beta\left[\cos\left(\beta-\frac{\alpha+\beta}{2}\right)-\sin\beta\sin\frac{\alpha+\beta}{2}\right]$$

$$=\sec\beta\cos\beta\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}$$

$\therefore$ The equation of chord is $\dfrac{x}{a}\cos\dfrac{\alpha-\beta}{2}-\dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2}=\cos\dfrac{\alpha+\beta}{2}$ ...... (20)

9. **Equation of tangent at $\theta$**.

As $\beta\to\alpha=\theta$, equation of chord $AB$ becomes a tangent at $P$ with parameter $\theta$.

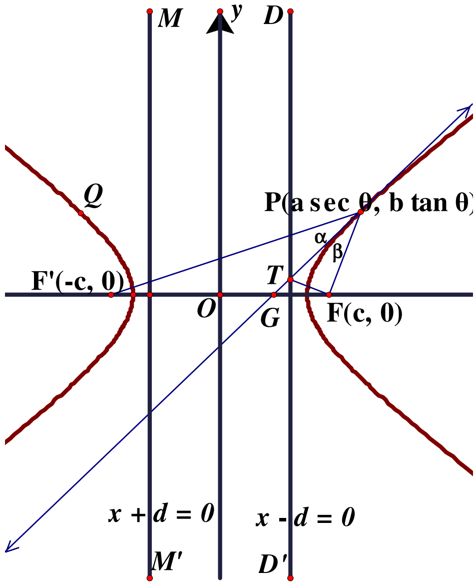
The equation of tangent at $\theta$ is given by $\dfrac{x}{a}-\dfrac{y}{b}\sin\theta=\cos\theta$

$\Rightarrow$ $\dfrac{x}{a}\sec\theta-\dfrac{y}{b}\tan\theta=1$ ...... (21)

If $(x_0, y_0)$ lies on the hyperbola, then $x_0=a\sec\theta$, $y_0=b\tan\theta$.

$\Rightarrow$ $\dfrac{x_0x}{a^2}-\dfrac{y_0y}{b^2}=1$ ...... (22)

10. Suppose the tangent at $P(a\sec\theta, b\tan\theta)$ cuts the directrix at $T$, then **$PT$ bisects $\angle FPF'$ and $\angle PFT = 90°$.**




Equation of tangent at $P$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

To find $T$: let $x=\frac{a}{e}$, then $y=b\cot\theta\left(\frac{\sec\theta}{e}-1\right)$

$T\left(\frac{a}{e},\frac{b}{e}\csc\theta(1-e\cos\theta)\right)$

$m_{TF}=\dfrac{\frac{b}{e}\csc\theta(1-e\cos\theta)}{\frac{a}{e}-ae}=\dfrac{b\csc\theta(1-e\cos\theta)}{a(1-e^2)}$

$m_{PF}=\dfrac{b\tan\theta}{a\sec\theta-ae}=\dfrac{b\sin\theta}{a(1-e\cos\theta)}$

$m_{TF}\times m_{PF}=\dfrac{b\csc\theta(1-e\cos\theta)}{a(1-e^2)}\times\dfrac{b\sin\theta}{a(1-e\cos\theta)}$

$=\dfrac{b^2}{a^2(1-e^2)}=\dfrac{c^2-a^2}{a^2(1-e^2)}=\dfrac{a^2e^2-a^2}{a^2(1-e^2)}=-1$

$\therefore TF\perp PF$

Let $\angle F'PT=\alpha$, $\angle TPF=\beta$, $PT$ is produced to cut $x$-axis at $G$, $\angle PGF=\phi$.

To find $G$, put $y=0$ in $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$ $\Rightarrow G=(a\cos\theta, 0)$.

$GF'=a\cos\theta+ae=a(\cos\theta+e)$, $FG=ae-a\cos\theta=a(e-\cos\theta)$

$PF'=\sqrt{(a\sec\theta+ae)^2+(b\tan\theta)^2}=\sqrt{a^2\sec^2\theta+2a^2e\sec\theta+a^2e^2+(a^2e^2-a^2)\tan^2\theta}$

$=\sqrt{a^2(\sec^2\theta-\tan^2\theta)+2a^2e\sec\theta+a^2e^2+a^2e^2(1+\tan^2\theta)}$

$=\sqrt{a^2+2a^2e\sec\theta+a^2e^2+a^2e^2\sec^2\theta}=\sqrt{(a+ae\sec\theta)^2}=a(1+e\sec\theta)$

$PF=\sqrt{(a\sec\theta-ae)^2+(b\tan\theta)^2}=a(e\sec\theta-1)$ $(\because e>1, \sec\theta>1, PF>0)$

Apply sine law on $\Delta PGF$, $\dfrac{FG}{PF}=\dfrac{\sin\beta}{\sin\phi}$

$\dfrac{a(e-\cos\theta)}{a(e\sec\theta-1)}=\dfrac{\sin\beta}{\sin\phi}\Rightarrow\cos\theta=\dfrac{\sin\beta}{\sin\phi}$ …… (23)

Apply sine law on $\Delta PGF'$, $\dfrac{F'G}{PF'}=\dfrac{\sin\alpha}{\sin\phi}$

$\dfrac{a(\cos\theta+e)}{a(1+e\sec\theta)}=\dfrac{\sin\alpha}{\sin\phi}\Rightarrow\cos\theta=\dfrac{\sin\alpha}{\sin\phi}$ …… (24)

Compare (21) and (22) $\Rightarrow\alpha=\beta$

$\therefore PT$ bisects $\angle FPF'$.

11. Let $P(a\sec\theta, b\tan\theta)$ be a point on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then **the difference of the focal distances is a constant = $2a$.**

$PF=\sqrt{(a\sec\theta-ae)^2+(b\tan\theta)^2}=a(e\sec\theta-1)$

$PF'=\sqrt{(a\sec\theta+ae)^2+(b\tan\theta)^2}=a(e\sec\theta+1)$

$|PF-PF'|=|a(e\sec\theta+1)-a(e\sec\theta-1)|=2a$

12. **Condition for tangency**.

Let $\ell x + my + n = 0$ be a tangent, then it is proportional to $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.

i.e. $\frac{\sec\theta}{a\ell} = -\frac{\tan\theta}{bm} = -\frac{1}{n}$

$\Rightarrow \sec\theta = -\frac{a\ell}{n}$, $\tan\theta = \frac{bm}{n}$

$\sec^2\theta - \tan^2\theta = 1 \Rightarrow \left(\frac{a\ell}{n}\right)^2 - \left(\frac{bm}{n}\right)^2 = 1 \Rightarrow (a\ell)^2 - (bm)^2 = n^2$

$\Rightarrow (bm)^2 + n^2 = (a\ell)^2$ …… (25)

13. **Equation of tangents, given slope *m***.

Let $y = mx + k$ be a tangent, then it is proportional to $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.

i.e. $\frac{\sec\theta}{am} = \frac{\tan\theta}{b} = -\frac{1}{k}$

$\Rightarrow \sec\theta = -\frac{am}{k}$, $\tan\theta = -\frac{b}{k}$

$\sec^2\theta - \tan^2\theta = 1 \Rightarrow \left(\frac{am}{k}\right)^2 - \left(\frac{b}{k}\right)^2 = 1 \Rightarrow (am)^2 - b^2 = k^2$

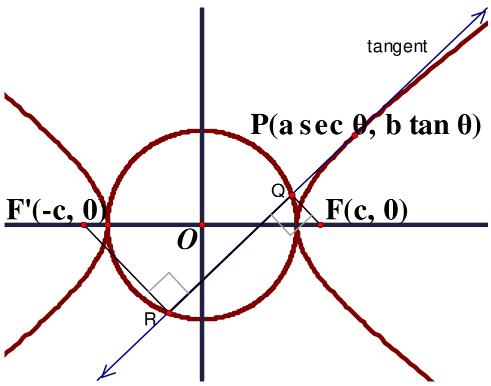
$\Rightarrow k = \pm\sqrt{a^2m^2 - b^2}$

$\therefore$ **Given a slope *m*, equation of tangents are** $y = mx \pm \sqrt{a^2m^2 - b^2}$ …… (26)

Note that when $a^2m^2 - b^2 \ge 0$, i.e. $m \ge \frac{b}{a}$ or $m \le -\frac{b}{a}$, two tangents can be found.

When $-\frac{b}{a} < m < \frac{b}{a}$, no tangents can be drawn.

14. The locus of the feet of perpendiculars from the foci to a tangent is **the auxiliary circle** with centre at $O(0, 0)$ and radius = $a$.




Equation of tangent: $y - mx = \pm\sqrt{a^2m^2 - b^2}$ …… (27)

The perpendicular line through $F$ is $my + x = ae$ ⋯ (28)

The perpendicular line through $F'$ is $my + x = -ae$ ⋯ (29)

$(27)^2 + (28)^2$: $(1 + m^2)(x^2 + y^2) = a^2e^2 - b^2 + a^2m^2$

$= c^2 - b^2 + a^2m^2$

$= a^2 + a^2m^2 = (1 + m^2)a^2$

$\Rightarrow x^2 + y^2 = a^2$

Similarly, eliminating $m$ from (27) and (29) gives $x^2 + y^2 = a^2$.

15. **The product of 2 perpendiculars from the foci is $b^2$.**

Using the formula (26) in section 13, equation of tangent is $y = mx \pm \sqrt{a^2m^2 - b^2}$

By distance formula, $QF$, $RF'$ $= \left| \frac{m(ae) \pm \sqrt{a^2m^2-b^2}}{\sqrt{1+m^2}} \times \frac{m(-ae) \pm \sqrt{a^2m^2-b^2}}{\sqrt{1+m^2}} \right|$

$= \left| \frac{a^2m^2 - b^2 - m^2a^2e^2}{1+m^2} \right| = \left| \frac{a^2m^2 - b^2 - m^2c^2}{1+m^2} \right|$, $ae = c$

$= \left| \frac{-b^2m^2 - b^2}{1+m^2} \right|$, $a^2 - c^2 = -b^2$

$= m^2$

16. The **asymptotes**. Clearly the hyperbola has no vertical asymptotes.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} - 1 = \frac{y^2}{b^2}$$

$$\Rightarrow y = \pm b\sqrt{\frac{x^2}{a^2} - 1} \Rightarrow \frac{y}{x} = \pm b\sqrt{\frac{1}{a^2} - \frac{1}{x^2}} \cdots\cdots (30)$$

Let $y = mx + k$ be an oblique asymptote.

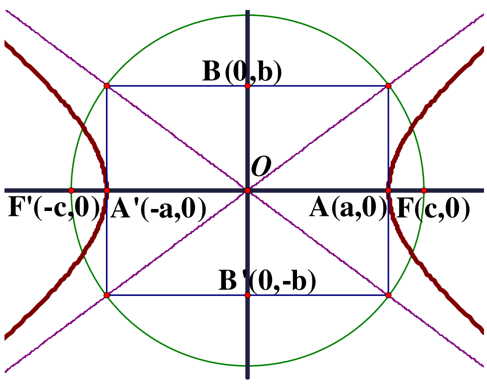
$$m = \lim_{x \to \infty} \frac{y}{x} = \lim_{x \to \infty} \left( \pm b\sqrt{\frac{1}{a^2} - \frac{1}{x^2}} \right) = \pm \frac{b}{a}$$

(Similarly, $m = \lim_{x \to -\infty} \frac{y}{x} = \pm \frac{b}{a}$)

$$k = \lim_{x \to \infty} (y - \frac{bx}{a}) = \lim_{x \to \infty} \left( \frac{b}{a}\sqrt{x^2 - a^2} - \frac{bx}{a} \right) = \frac{b}{a} \lim_{x \to \infty} \left( \frac{x^2 - a^2 - x^2}{\sqrt{x^2 - a^2} + x} \right) = 0$$

$$k = \lim_{x \to \infty} (y + \frac{bx}{a}) = \lim_{x \to \infty} \left( -\frac{b}{a}\sqrt{x^2 - a^2} + \frac{bx}{a} \right) = \frac{b}{a} \lim_{x \to \infty} \left( \frac{x^2 - x^2 + a^2}{\sqrt{x^2 - a^2} + x} \right) = 0$$

$\therefore$ The two asymptotes are $y = \frac{b}{a}x$ …… (31) and $y = -\frac{b}{a}x$ …… (32)




$AA'$ is called the **transverse axis** ($2a$).

$BB'$ is called the **conjugate axis** ($2b$), it has no real intersection with the curve.

17. **Rectangular hyperbola**

If $a = b$, the asymptotes are the lines $x + y = 0$ and $x - y = 0$, which are at right angles.

The hyperbola is then said to be **rectangular** or **equilateral**.

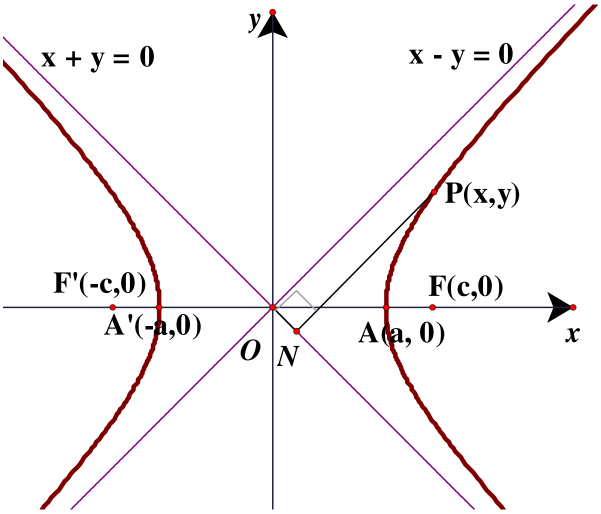
To find the eccentricity, $a^2 + b^2 = c^2 = a^2e^2$

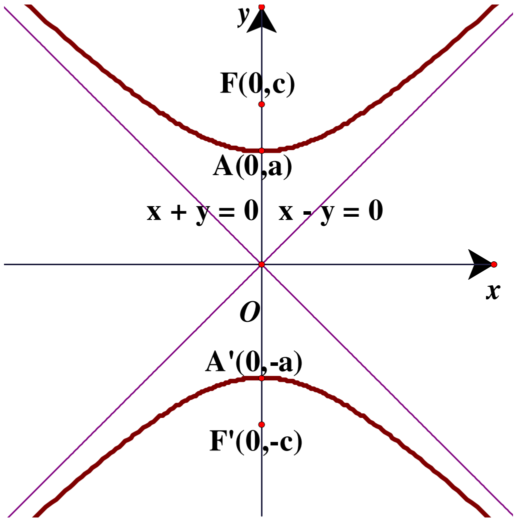
$2a^2 = a^2e^2$

$e = \sqrt{2}$

The equation may be

$x^2 - y^2 = a^2 \cdots\cdots (33)$ or $y^2 - x^2 = a^2 \cdots\cdots (34)$




Let $P(x, y)$ be a point on $x^2 - y^2 = a^2 \cdots\cdots (33)$

Let $N$ be the foot of perpendicular from $P$ on the line $x + y = 0$.

$x = ON \cos 45° + PN \sin 45° = \dfrac{ON + PN}{\sqrt{2}}$

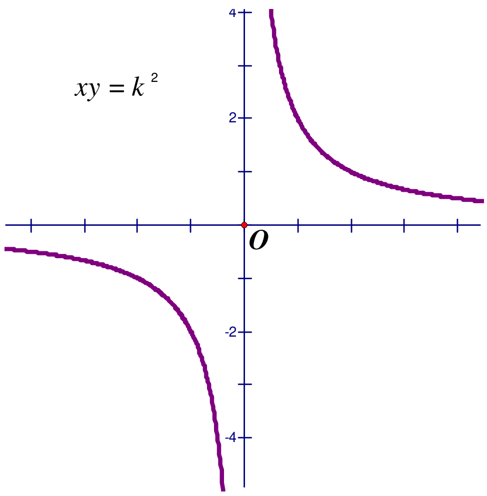
$y = PN \sin 45° - ON \cos 45° = \dfrac{PN - ON}{\sqrt{2}}$

(33): $x^2 - y^2 = a^2 \Rightarrow \frac{1}{2}(ON + PN)^2 - \frac{1}{2}(PN - ON)^2 = a^2$

$2\ ON \cdot PN = a^2$

If we take the asymptote $ON$ as the new $X$-axis, and the other asymptote ($x - y = 0$) as the new $Y$-axis, then the new equation of the rectangular hyperbola, referred to its asymptotes, is in the form: $XY = k^2$, where $k^2 = \frac{1}{2}a^2$.

With this system, the asymptotes are given by $X = 0$ or $Y = 0$.



18. **Parametric equations**

$\begin{cases} x = kt \\ y = \dfrac{k}{t} \end{cases}$ $\cdots\cdots$ (35) are the parametric equations of $xy = k^2$.

19. **Chord, tangent and normal**.

Let $A\left(kt_1, \dfrac{k}{t_1}\right)$, $B\left(kt_2, \dfrac{k}{t_2}\right)$ be two points on the rectangular hyperbola $xy = k^2$.

Equation of chord $AB$ is given by: $\dfrac{y - \frac{k}{t_2}}{x - kt_2} = \dfrac{\frac{k}{t_1} - \frac{k}{t_2}}{kt_1 - kt_2} = \dfrac{k(t_2 - t_1)}{kt_1t_2(t_1 - t_2)} = -\dfrac{1}{t_1t_2}$

$t_1t_2y - kt_1 + x - kt_2 = 0$

$\therefore$ The **equation of chord** is: $x + t_1t_2y - k(t_1 + t_2) = 0$ $\cdots\cdots$ (36)

As $B \to A$, the **equation of tangent** at $A$ is $x + t^2y - 2kt = 0$ $\cdots\cdots$ (37)

**Equation of normal** is given by: $\dfrac{y - \frac{k}{t}}{x - kt} = t^2$

which is equivalent to $t^3x - ty - kt^4 + k = 0$ $\cdots\cdots$ (38)

20. **Example 2** Find the parameter of the point in which the normal at $t$ meet the curve again.

Let the required parameter be $t_2$.

Compare the normal $t^3x - ty - kt^4 + k = 0$ $\cdots\cdots$ (38) and chord: $x + tt_2y - k(t + t_2) = 0$ $\cdots\cdots$ (36)

Since they are identical,

$\therefore \dfrac{t^3}{1} = \dfrac{-t}{tt_2} = \dfrac{-kt^4 + k}{-k(t + t_2)}$

$\Rightarrow t_2 = -\dfrac{1}{t^3}$

**Example 3** Find the locus of mid points of chords $xy = k^2$ which are parallel to **the diameter** $\boldsymbol{y = mx}$.

Let the chord be $x + t_1t_2y - k(t_1 + t_2) = 0$ $\cdots\cdots$ (36)

It is parallel to $y = mx \Rightarrow m = -\dfrac{1}{t_1t_2}$.

The mid-point is $x = \dfrac{k}{2}(t_1 + t_2)$ $\cdots\cdots$ (39), $y = \dfrac{k}{2}\left(\dfrac{1}{t_1} + \dfrac{1}{t_2}\right) = \dfrac{k}{2} \cdot \dfrac{t_1 + t_2}{t_1t_2}$ $\cdots\cdots$ (40)

(40) $\div$ (39): $\dfrac{y}{x} = \dfrac{1}{t_1t_2} = -m$

$\therefore$ The locus is $y = -mx$. It is called **the conjugate diameter**.

**Example 4** If $(x_0, y_0)$ lie on $xy = k^2$, then $x_0 = kt$, $y_0 = \dfrac{k}{t}$.

The equation of tangent at $t$ is : $x + t^2y - 2kt = 0$

$\dfrac{kx}{t} + (kt)y - 2k^2 = 0$

$\dfrac{1}{2}(xy_0 + x_0y) = k^2$

**Example 5** If $PP'$ is a diameter of a rectangular hyperbola and the normal at $P$ meet the curve again at $Q$, show that $PQ$ subtends a right angle at $P'$.

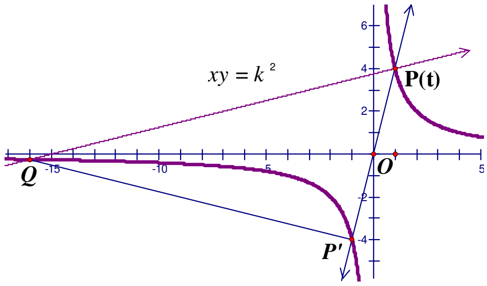
$PP'$ passes through $O(0, 0)$, $PQ$ = normal at $P$.

Try to show that $\angle PP'Q = 90°$

Let the parameter of $P$ be $t$.

Then since $PP'$ passes through $O$, parameter of $P' = -t$

By example 2, parameter of $Q$ is $-\frac{1}{t^3}$.




$$m_{PP'} \times m_{QP'} = \frac{1}{t^2} \times \frac{-kt^3 - \frac{k}{t}}{-\frac{k}{t^3} + kt} = \frac{1}{t^2} \times \left(-t^2\right) = -1$$

$\therefore PP' \perp QP'$

21. There are **at most 4 normals pass through a given point ($p$, $q$).**

The normal at $\left(kt, \frac{k}{t}\right)$ is $t^3x - ty - kt^4 + k = 0$

It passes through $(p, q)$: $t^3p - tq - kt^4 + k = 0$

$\Rightarrow kt^4 - pt^3 + qt - k = 0$

It is a polynomial equation of degree 4, which has, in general at most 4 (real) roots in $t$ ($t_1$, $t_2$, $t_3$, $t_4$).

$t_1 t_2 t_3 t_4 = -1$

If $t_1$, $t_2$, $t_3$, $t_4$ are real, 3 lies on one branch and one on the other, (say) $t_1$, $t_2$, $t_3$ is positive and $t_4$ is negative or $t_1$, $t_2$, $t_3$ is negative and the $t_4$ is positive.