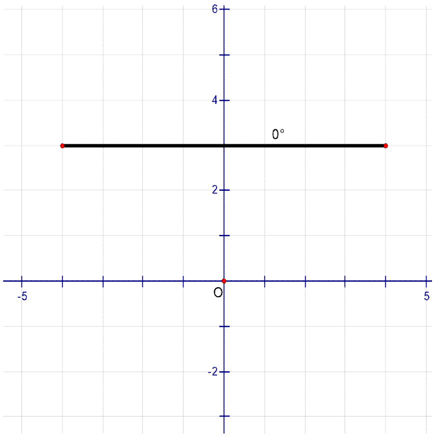
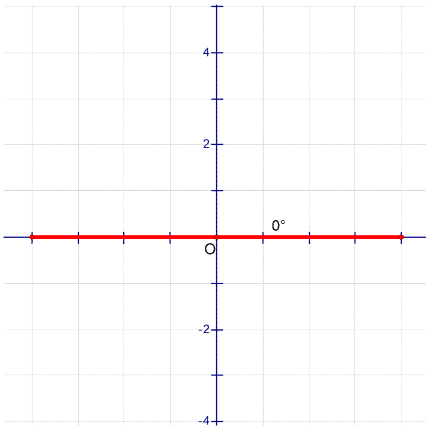
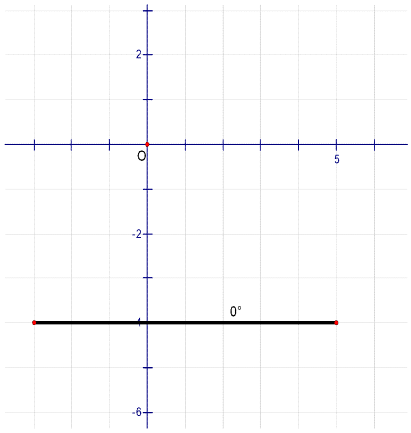
Inclination of a Straight Line

Created by Mr. Francis Hung on 20091202 Last updated: February 27, 2023

Given a straight line *L*. If the line is parallel to *x*-axis, then the inclination of a straight line is 0°.



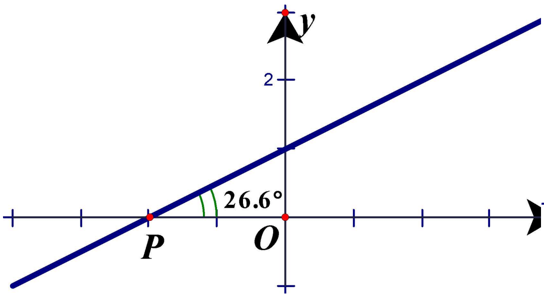


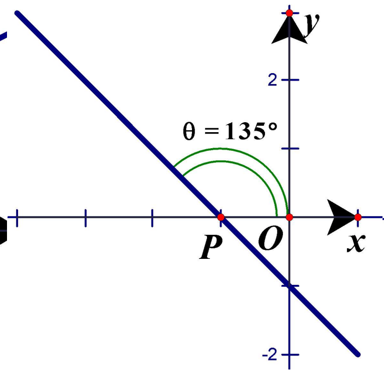


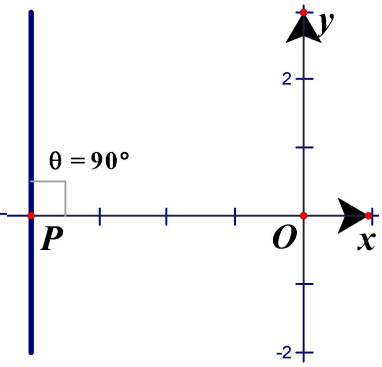
If the line *L* is not parallel to *x*-axis, then *L* will intersect *x*-axis at a point *P*. The angle at which the line *L* makes with positive *x*-axis, measured in anti-clockwise direction, is the **inclination** of the straight line *L*.

If the inclination of the straight line *L* is θ, then $0° \le \theta < 180°$.

The following figures show different cases:







Slope of a Straight Line

If the inclination of a straight line is θ and θ ≠ 90°, then the slope of the straight line is defined as: $m = \tan\theta$.

If the inclination of a straight line is **90°**, then **the slope** of the straight line **is undefined**.

In particular, θ = 0°, $m = \tan 0° = 0$. i.e. **the slope of a line parallel to *x*-axis is 0.**

θ = 63.4°, $m = \tan 63.4° = 2$. i.e. a line starting from lower left hand to upper right hand has a **positive slope**.

θ = 135°, $m = \tan 135° = -1$. i.e. a line starting from upper left hand to lower right hand has a **negative slope**.

Given that a line passes through two points $A(x_1, y_1)$, $B(x_2, y_2)$ and let the inclination of AB be θ.

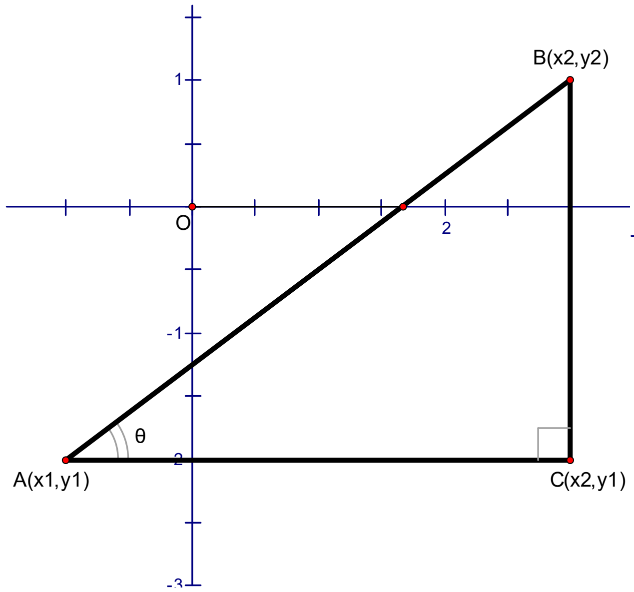
If θ ≠ 90°, then $x_1 \neq x_2$. Let C be (x_2, y_1). Join AC, BC. Then $\angle ACB = 90°$.

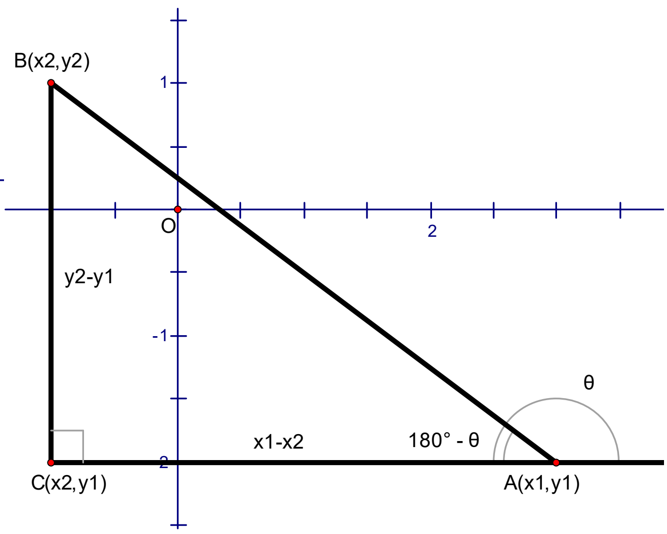
If θ < 90°, then $\tan\theta = \dfrac{BC}{AC} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

If θ > 90°, then $\tan(180° - \theta) = \dfrac{BC}{AC} = \dfrac{y_2 - y_1}{x_1 - x_2}$.

$-\tan\theta = -\dfrac{y_2 - y_1}{x_2 - x_1}$

$\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$





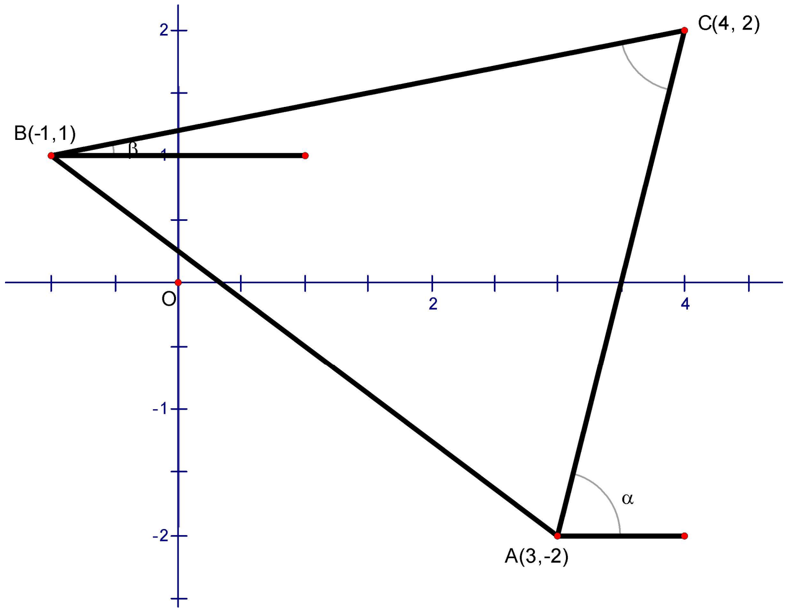
In both cases, $m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

Example 1 Find the slope and the inclination of the line passes through $A(3, -2)$, $B(-1, 1)$.

$m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1-(-2)}{-1-3} = -\dfrac{3}{4}$

θ = 143.1°

Example 2 Given $A(3, -2)$, $B(-1, 1)$, $C(4, 2)$. Find $\angle ACB$.



$m_{AC} = \tan\alpha = \dfrac{2-(-2)}{4-3} = 4$, $\alpha = 75.96°$

$m_{BC} = \tan\beta = \dfrac{2-1}{4-(-1)} = \dfrac{1}{5}$, $\beta = 11.31°$

$\angle ACB = \alpha - \beta = 75.96° - 11.31° = 64.65°$ (ext. $\angle$ of Δ)

Parallel Lines

Two lines L_1 and L_2 are parallel if their inclinations are equal. That is to say, if the inclination of L_1 is α and the inclination of L_2 is β, then $L_1 // L_2$ if and only if $\alpha = \beta$.

Suppose $\alpha \neq 90°$, $\beta \neq 90°$, then $L_1 // L_2$ if and only if $m_{L1} = m_{L2}$.

$L_1 // L_2 \Leftrightarrow \alpha = \beta \Leftrightarrow \tan\alpha = \tan\beta \Leftrightarrow m_{L1} = m_{L2}$

Example 3 Given $A(-4, -2)$, $B(b, b-2)$, $C(3, 4)$. If A, B and C lie on the same straight line, find b.

$AB // AC \Rightarrow m_{AB} = m_{AC}$

$$\Rightarrow \frac{b-2-(-2)}{b-(-4)} = \frac{4-(-2)}{3-(-4)}$$

$$\Rightarrow \frac{b}{b+4} = \frac{6}{7}$$

$7b = 6b + 24$

$b = 24$

Example 4 Given $A(-5,6)$, $B(-2,4)$, $C(x, y)$ and $D(-3,3)$. If $ABCD$ forms a parallelogram, find x and y.

$AB // CD \Rightarrow \dfrac{y-3}{x+3} = \dfrac{4-6}{-2+5} \Rightarrow \dfrac{y-3}{x+3} = -\dfrac{2}{3} \Rightarrow 3y - 9 = -2x - 6 \Rightarrow 2x + 3y - 3 = 0$ …… (1)

$AD // BC \Rightarrow \dfrac{y-4}{x+2} = \dfrac{3-6}{-3+5} \Rightarrow \dfrac{y-4}{x+2} = -\dfrac{3}{2} \Rightarrow 2y - 8 = -3x - 6 \Rightarrow 3x + 2y - 2 = 0$ …… (2)

$3\times(1) - 2\times(2)$: $5y - 5 = 0 \Rightarrow y = 1$

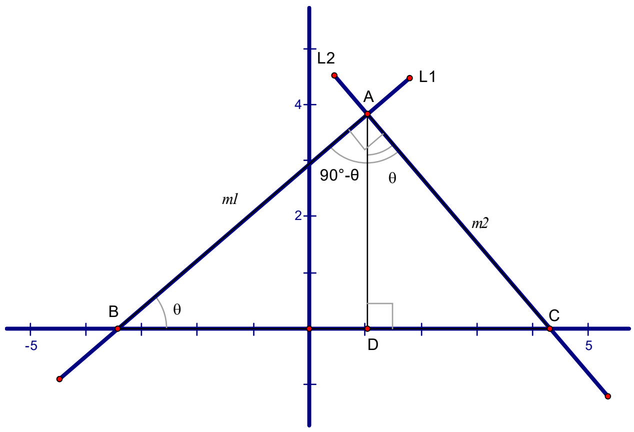
$3\times(2) - 2\times(1)$: $5x = 0 \Rightarrow x = 0$

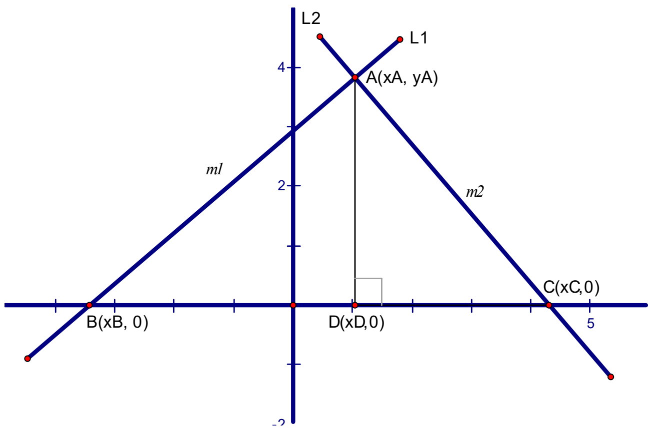
Perpendicular Lines

Suppose the inclination of a line L_1 is α and the inclination of another line L_2 is β. The two lines are perpendicular if and only if $|\beta - \alpha| = 90°$.

If $\alpha \neq 0°$ and $\beta \neq 0°$, then $L_1 \perp L_2$ if and only if $m_{L1} \cdot m_{L2} = -1$

Proof: **Method 1**





Draw a line $AD \perp BC$.

Let $\angle ABD = \theta$

then $\angle BDA = 90°$ ($\because AD \perp BC$)

$\angle BAD = 180° - \theta - 90°$ (∠ sum of Δ)

$= 90° - \theta$

$\angle CAD = \angle BAC - \angle BAD$

$= 90° - (90° - \theta)$ ($\because L_1 \perp L_2$)

$= \theta$

$\therefore \angle ABD = \angle CAD = \theta$

$\angle ADB = \angle ADC = 90°$ (by construction)

$\therefore \Delta ABD \sim \Delta CAD$ (equiangular)

$\frac{CD}{AD} = \frac{AD}{BD}$ (corr. sides, ~Δs)

$\frac{AD \times AD}{BD \times CD} = 1$ …… (1)

Let the coordinates of $A(x_A, y_A)$, $B(x_B, 0)$, $C(x_C, 0)$, $D(x_D, 0)$

$m_1 \times m_2$

$= \frac{y_A - 0}{x_A - x_B} \cdot \frac{0 - y_A}{x_C - x_A}$

$= \frac{AD}{BD} \cdot \frac{-AD}{CD}$

$= -\frac{AD}{BD} \cdot \frac{AD}{CD}$

$= -1$

$\therefore m_1 \times m_2 = -1$

Conversely, if $m_1 \times m_2 = -1$,

$\frac{y_A - 0}{x_A - x_B} \cdot \frac{0 - y_A}{x_C - x_A} = -1$

$\frac{AD}{BD} \cdot \frac{-AD}{CD} = -1$

$\frac{AD \times AD}{BD \times CD} = 1$

$\frac{CD}{AD} = \frac{AD}{BD}$

$\angle ADB = \angle ADC = 90°$ (by construction)

$\Delta ABD \sim \Delta CAD$ (ratio of 2 sides, included ∠)

$\angle ABD = \angle CAD$ (corr. ∠s, ≅ Δs)

$\angle ABD + \angle BAD + 90° = 180°$ (∠ sum of Δ)

$\angle CAD + \angle BAD = 90°$

$\angle BAC = 90°$

$\therefore L_1 \perp L_2$.

Method 2

Without loss of generality, assume $\beta > \alpha$.

As shown in the diagram, $\beta = 90° + \alpha$ (ext. ∠ of Δ)

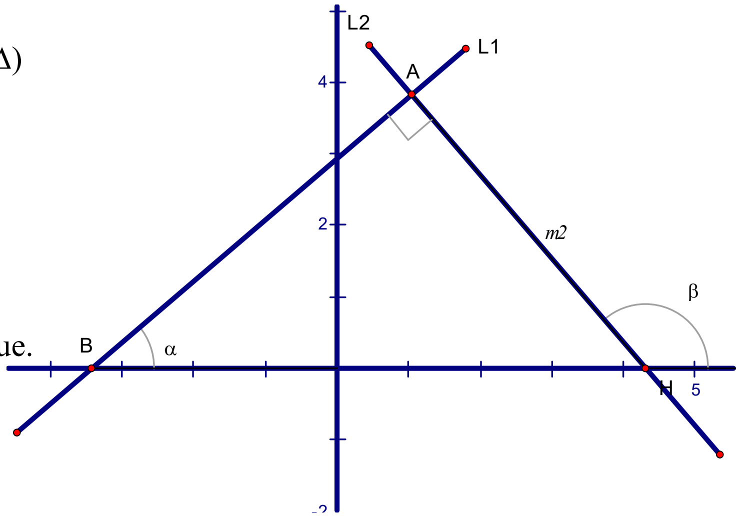
$\tan\beta = \tan(90° + \alpha)$

$\tan\beta = -\frac{1}{\tan\alpha}$

$\tan\alpha \tan\beta = -1$

$m_1 m_2 = -1$

It can be easily proved that the converse is also true.



Example 5 Given $A(2, 3)$, $B(5, 2)$ and $C(4, y)$

(a) If $AB \perp BC$, find the value of y.

(b) Show that ΔABC is isosceles.

(a) $m_{AB} \times m_{BC} = -1$

$\frac{3-2}{2-5}\cdot\frac{y-2}{4-5} = -1$

$\frac{1}{-3}\cdot\frac{y-2}{-1} = -1$

$y - 2 = -3$

$y = -1$

(b) $AB = \sqrt{(2-5)^2 + (3-2)^2} = \sqrt{10}$

$BC = \sqrt{(5-4)^2 + (2--1)^2} = \sqrt{10}$

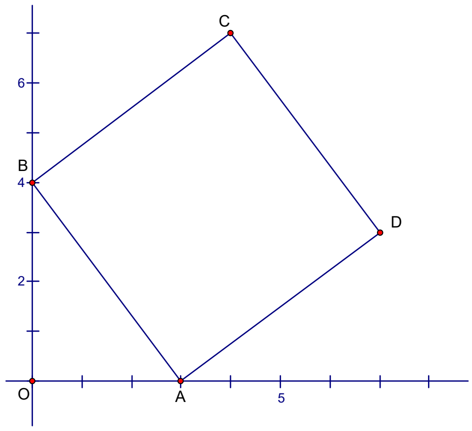
$AB = BC$

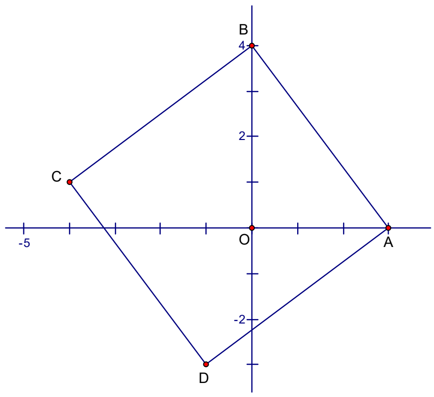
∴ ΔABC is isosceles.

Example 6 Given $A(3,0)$, $B(0,4)$. If $ABCD$ form a square, find C.

There are two possible cases.

Case 1 C, D lies on the upper right hand corner. **Case 2** C, D lies on the lower left hand corner.





Method 1 Let $C = (x, y)$

$AB = \sqrt{(3-0)^2 + (0-4)^2} = 5 = BC = \sqrt{(x-0)^2 + (y-4)^2} \Rightarrow x^2 + (y-4)^2 = 25$ …………(1)

$AB \perp BC \Rightarrow m_{AB} \cdot m_{BC} = -1$

$\frac{0-4}{3-0} \cdot \frac{y-4}{x-0} = -1 \Rightarrow 4y - 16 = 3x \Rightarrow x = \frac{4y-16}{3}$ …………(2)

Sub. (2) into (1): $\left(\frac{4y-16}{3}\right)^2 + (y-4)^2 = 25$

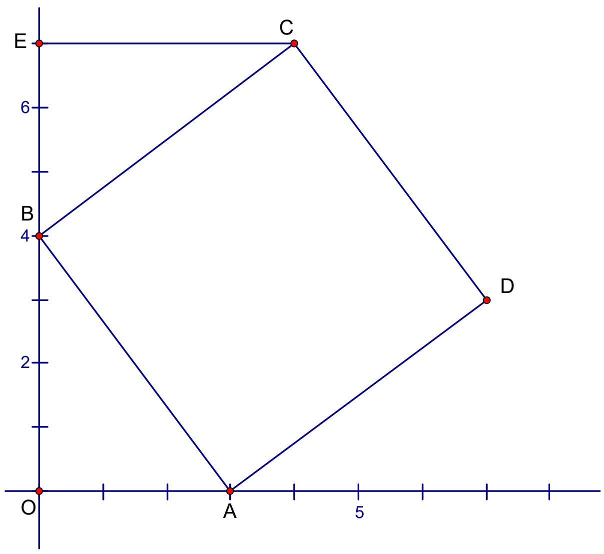
$16y^2 - 128y + 256 + 9(y^2 - 8y + 16) = 225$

$25y^2 - 200y + 175 = 0$

$25(y-1)(y-7) = 0$

$y = 1$ or $y = 7 \Rightarrow x = -4$ or $x = 4 \Rightarrow C = (-4, 1)$ or $(4, 7)$

Method 2



Suppose C lies on the upper right hand corner.

Let $C = (x, y)$. $x > 0$, $y > 0$.

Let $E = (0, y)$, let $\angle OBA = \theta$

$\angle ABC = 90°$ (property of a square)

$\angle CBE = 180° - 90° - \theta$ (adj. ∠s on st. line)

$= 90° - \theta$

$\angle BCE = 90° - (90° - \theta) = \theta$ (∠s sum of Δ)

$AB = BC$ (property of a square)

$\therefore \Delta AOB \cong \Delta BOC$ (ASA)

$BE = y - 4 = OA = 3$ (corr. sides ≅ Δs)

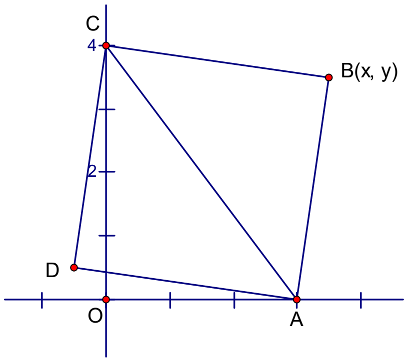
$y = 7$

$CE = x = OB = 4$ (corr. sides ≅ Δs)

$\therefore C = (4, 7)$

When C lies on the lower left hand corner, the work is similar.

Example 7 Given $A(3,0)$, $C(0,4)$. If $ABCD$ form a square, find B.



Method 1 Let $B = (x, y)$

$AB \perp BC \Rightarrow m_{AB} \cdot m_{BC} = -1$

$\frac{y-0}{x-3} \cdot \frac{y-4}{x-0} = -1 \Rightarrow y^2 - 4y = -x^2 + 3x$

$x^2 + y^2 - 3x - 4y = 0$ ⋯⋯⋯⋯(1)

$AB = BC \Rightarrow \sqrt{(x-3)^2 + y^2} = \sqrt{x^2 + (y-4)^2}$

$x^2 - 6x + 9 + y^2 = x^2 + y^2 - 8y + 16$

$6x - 8y + 7 = 0$

$y = \frac{6x+7}{8}$ ⋯⋯⋯⋯(2)

Sub. (2) into (1)

$x^2 + \left(\frac{6x+7}{8}\right)^2 - 3x - 4\left(\frac{6x+7}{8}\right) = 0$

$64x^2 + (36x^2 + 84x + 49) - 192x - (192x + 224) = 0$

$100x^2 - 300x - 175 = 0$

$4x^2 - 12x - 7 = 0$

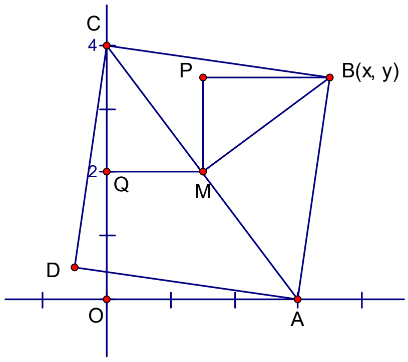
$(2x + 1)(2x - 7) = 0$

$x = -\frac{1}{2}$ or $\frac{7}{2}$

$y = \frac{1}{2}$ or $\frac{7}{2}$

$B(-\frac{1}{2}, \frac{1}{2})$ or $(\frac{7}{2}, \frac{7}{2})$

Method 2 Suppose B lies on the upper right hand corner.



M = mid point of AC = (1.5, 2)

Let $Q = (0, 2)$, $P = (1.5, y)$

Then $\angle CQM = \angle BPM = 90°$ (by construction)

$CM = BM$ (diagonal of a square)

$\angle CMQ = 90° - \angle CMP = \angle BMC - \angle CMP = \angle BMP$

$\Delta CQM \cong \Delta BPM$ (AAS)

$BP = x - 1.5 = CQ = 4 - 2 = 2$ (corr. sides ≅ Δs)

$x = 3.5$

$PM = y - 2 = QM = 1.5$ (corr. sides ≅ Δs)

$y = 3.5 \Rightarrow B = (3.5, 3.5)$

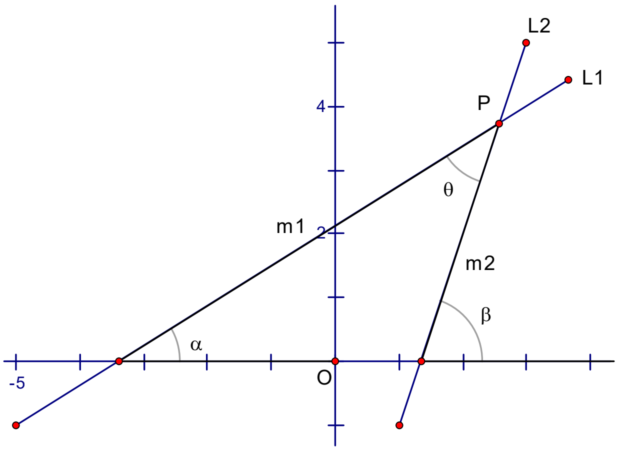
The case where B lies on the lower left hand corner is similar.

Exercise 1 Given $A(0, 0)$, B, $C(8, 6)$ and D form a rhombus with each side = 13. Find B and D.

[Ans. (–3.2, 12.6), (11.2, –6.6)]

Angle between two straight Lines

Given two non-parallel straight lines L_1 and L_2. Suppose they intersect at a point P. Let α and β be the inclinations of L_1 and L_2 respectively. Let their respective slopes be m_1 and m_2. Four angles are formed at P. If they are perpendicular, then the angle between L_1 and L_2 is 90°. Otherwise, the angle between the two lines ids defined as the **acute angle**, θ, between them.



Without loss of generality, assume $\beta > \alpha$, then $\theta = \beta - \alpha$ (ext. ∠ of Δ)

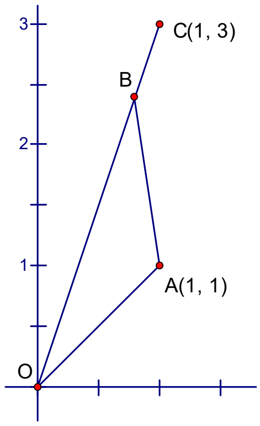
$\tan\theta = \tan(\beta - \alpha)$

$$\tan\theta = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta}$$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

If we don't know whether α or β is bigger, then $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$.

Example 8 Given $O(0, 0)$, $A(1, 1)$, $C(1, 3)$. B is a point on OC such that $OA = AB$. Find the slope of AB.



slope of OA = 1, slope of OC = slope of OB = 3

Let $\angle AOB = \theta = \angle ABO$ (base ∠s, isos. Δ)

$$\tan\theta = \left|\frac{3-1}{1+3}\right| = \frac{1}{2}$$

Let the slope of AB be m

$$\frac{1}{2} = \left|\frac{m-3}{1+3m}\right|$$

$1 + 3m = 2(m - 3)$ or $1 + 3m = 2(3 - m)$

$\Rightarrow m = -7$ or 1 (rejected)

$\therefore m = -7$