Rotation of axis example

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Given the central conics: $Ax^2 + Bxy + Cy^2 = D$, $A \neq C$, $B \neq 0$, $D \neq 0$

Perform the rotation of axes: $\begin{cases} x = x_1 \cos\theta - y_1 \sin\theta \\ y = x_1 \sin\theta + y_1 \cos\theta \end{cases}$, where $\theta = \frac{1}{2}\tan^{-1}\frac{B}{A-C}$.

Prove that after the transformation, the equation is in the form $ax_1^2 + by_1^2 = D$.

Proof:

$$\tan 2\theta = \frac{B}{A-C}$$

$$A(x_1 \cos\theta - y_1 \sin\theta)^2 + B(x_1 \cos\theta - y_1 \sin\theta)(x_1 \sin\theta + y_1 \cos\theta) + C(x_1 \sin\theta + y_1 \cos\theta)^2 = D$$

$$\left(A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta\right)x_1^2 + \left[-2A\sin\theta\cos\theta + B\left(\cos^2\theta - \sin^2\theta\right) + 2C\sin\theta\cos\theta\right]x_1y_1 + \left(A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta\right)y_1^2 = D$$

$$\left(A\frac{1+\cos 2\theta}{2} + \frac{B\sin 2\theta}{2} + C\frac{1-\cos 2\theta}{2}\right)x_1^2 + \left[(C-A)\sin 2\theta + B\cos 2\theta\right]x_1y_1 + \left(A\frac{1-\cos 2\theta}{2} - \frac{B\sin 2\theta}{2} + C\frac{1+\cos 2\theta}{2}\right)y_1^2 = D$$

$$\left[A + C + B\sin 2\theta + (A-C)\cos 2\theta\right]x_1^2 + 2\left[(C-A)\sin 2\theta + B\cos 2\theta\right]x_1y_1 + \left[A + C - B\sin 2\theta + (C-A)\cos 2\theta\right]y_1^2 = 2D$$

$$\left[A + C + \frac{B^2}{\sqrt{(A-C)^2 + B^2}} + \frac{(A-C)^2}{\sqrt{(A-C)^2 + B^2}}\right]x_1^2 + 2\left(\frac{B(C-A)}{\sqrt{(A-C)^2 + B^2}} + \frac{B(A-C)}{\sqrt{(A-C)^2 + B^2}}\right)x_1y_1 + \left[A + C - \frac{B^2}{\sqrt{(A-C)^2 + B^2}} - \frac{(A-C)^2}{\sqrt{(A-C)^2 + B^2}}\right]y_1^2 = 2D$$

$$\left[A + C + \sqrt{(A-C)^2 + B^2}\right]x_1^2 + \left[A + C - \sqrt{(A-C)^2 + B^2}\right]y_1^2 = 2D$$

$$\left[\frac{A + C + \sqrt{(A-C)^2 + B^2}}{2}\right]x_1^2 + \left[\frac{A + C - \sqrt{(A-C)^2 + B^2}}{2}\right]y_1^2 = D$$

It is in the form $ax_1^2 + by_1^2 = D$.